$$F_x = -\frac{\partial U}{\partial x}$$

$$M_{\rm int} = \frac{4\pi}{3}\rho(t) r(t)^3$$

$$= \frac{4\pi}{3}\rho_0 r_0^3$$

$$\mathrm{KE} + \mathrm{GPE} = \frac{1}{2}mv^2 - \frac{GM_{\rm int}m}{r} = -\frac{1}{2}mkc^2\tilde{\omega}^2$$

$$r(t) \equiv \tilde{\omega} a(t)$$

$\tilde{\omega}$	comoving coordinate	meters
$a(t)$	scale factor	no units

$$\tilde{\omega}^2 \left(\frac{da}{dt}\right)^2 - \frac{8\pi}{3} G\rho\tilde{\omega}^2 a^2 = -kc^2\tilde{\omega}^2$$

$$\left(\frac{da}{dt}\right)^2 - \frac{8\pi}{3} G\rho a^2 = -kc^2$$

$$H(t) = \frac{1}{a(t)}\frac{da}{dt}$$

$$\left[H(t)^2 - \frac{8\pi}{3} G\rho(t)\right] a(t)^2 = -kc^2$$

$$\left[H_0^2 - \frac{8\pi}{3} G\rho_0\right] a_0^2 = -kc^2$$

$$\Omega_M \equiv \frac{\rho_0}{\rho_{\rm crit}}$$

$$\left(\frac{da}{dt}\right)^2 - \frac{8\pi}{3} G\rho(t) a(t)^2 = -kc^2$$

$$
\begin{aligned}
\rho_0 {a_0}^3 &= \rho(t) a(t)^3 \\
\rho_0 1^3 &= \rho(t) a(t)^3 \\
\frac{\rho_0}{a(t)^2} &= \rho(t) a(t)^2
\end{aligned}
$$

$$
\left(\frac{da}{dt}\right)^2 - \left(\frac{8\pi}{3} G\rho_0\right) \frac{1}{a(t)} = -kc^2
$$

$$
a(x) = \frac{1}{2} \frac{\Omega_M}{\Omega_M - 1} \left(1 - \cos(x)\right)
$$

$$
t(x) = \frac{1}{2H_0} \frac{\Omega_M}{(\Omega_M - 1)^{3/2}} \left(x - \sin(x)\right)
$$

$$
a(t) = \left(6\pi G\rho_0\right)^{1/3} \, t^{2/3}
$$

$$
a(x) = \frac{1}{2} \frac{\Omega_M}{1 - \Omega_M} \left(\cosh(x) - 1\right)
$$

$$
t(x) = \frac{1}{2H_0} \frac{\Omega_M}{(1 - \Omega_M)^{3/2}} \left(\sinh(x) - x\right)
$$

$$
z = \frac{\lambda_{\text{obs}} - \lambda_{\text{emit}}}{\lambda_{\text{emit}}} = \frac{a_{\text{obs}} - a_{\text{emit}}}{a_{\text{emit}}}
$$

$$
q(t) \equiv \frac{a(t) \left(d^2a/dt^2\right)}{\left(da/dt\right)^2}
$$

$$
q(t) = \frac{1}{2} \, \Omega_M(t)
$$

$$
(ds)^2 = (c\, dt)^2 - a(t)^2 \left(\frac{d\tilde{\omega}^2}{1 - k\tilde{\omega}^2}\right)
$$

$$
c^2\, dt^2 = a(t)^2 \left(\frac{d\tilde{\omega}^2}{1 - k\tilde{\omega}^2}\right)
$$

$$c\,\frac{dt}{a(t)} \quad = \quad \frac{d\tilde{\omega}}{\sqrt{1-k\tilde{\omega}^2}}$$

$$c\int_0^{t_{\rm obs}}\frac{dt}{a(t)} \quad = \quad \int_0^{\tilde{\omega}_e}\frac{d\tilde{\omega}}{\sqrt{1-k\tilde{\omega}^2}}$$

$$\tilde{\omega}(z) \quad = \quad \frac{2c}{H_0}\left(\frac{1}{1+z}\right)\frac{1}{\Omega_M^2}\left(\Omega_M z-(2-\Omega_M)\left[\sqrt{1+\Omega_M z}-1\right]\right)$$

$$f \quad \stackrel{?}{=} \quad \sqrt{\frac{L}{4\pi\tilde{\omega}^2}}$$

$$f_{\rm photon} \quad = \quad \sqrt{\frac{L}{4\pi\tilde{\omega}^2(1+z)}}$$

$$D_{L\ \rm photon} \quad = \quad \sqrt{\frac{L}{4\pi f_{\rm photon}}} \quad = \quad \tilde{\omega}\sqrt{1+z}$$

$$f_{\rm energy} \quad = \quad \sqrt{\frac{L}{4\pi\tilde{\omega}^2(1+z)(1+z)}}$$

$$D_{L\ \rm energy} \quad = \quad \sqrt{\frac{L}{4\pi f_{\rm energy}}} \quad = \quad \tilde{\omega}(1+z)$$

$$\text{total energy} \quad = \quad \frac{1}{2}mv^2-\frac{GM_{\rm int}m}{r}-\frac{1}{6}\Lambda mc^2r^2 \quad = \quad -\frac{1}{2}mkc^2\tilde{\omega}^2$$

$$q_0 \quad = \quad \frac{1}{2}\Omega_M\,-\,\frac{\Lambda c^2}{3H_0^2}$$

$$\rho \quad = \quad \text{density}\ \left(\frac{\rm kg}{\rm m^3}\right) \qquad \Omega_M \quad = \quad \frac{\rho}{\rho_c}\ \text{(no units)}$$

$$\Lambda \quad = \quad \text{lambda} \left(\frac{1}{\text{m}^2}\right) \qquad \Omega_\Lambda \quad = \quad \frac{\Lambda c^2}{3H_0^2} \text{ (no units)}$$

$$\Omega_{\text{tot}} \quad = \quad \Omega_M + \Omega_\Lambda$$

$$\Omega_K \quad \equiv \quad 1 - \Omega_{\text{tot}} \quad = \quad 1 - (\Omega_M + \Omega_\Lambda)$$

$$D_M \quad = \quad \frac{c}{H_0\sqrt{|\Omega_K|}} \text{ sinn} \left[\sqrt{|\Omega_K|} \int_0^{z1} \left[(1+z)^2(1+\Omega_M z) - z(2+z)\Omega_\Lambda\right]^{-1/2} dz\right]$$

$$\begin{aligned} D_A &= D_M \left(\frac{1}{1+z}\right) \\ \text{photon} - \text{based} \quad D_L &= D_M\sqrt{1+z} \\ \text{energy} - \text{based} \quad D_L &= D_M(1+z) \end{aligned}$$