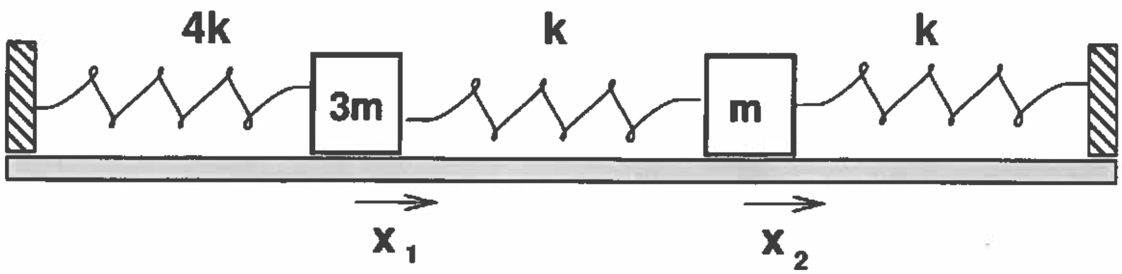
1:55 - 2:00



Problem 1

Two blocks of mass **3m** and **m** are connected by three springs, with force constants **4k**, **k** and **k**, as shown. Ignore all friction.

a. Write equations for the force on each block.
b. Convert those equations into a single matrix equation.
c. Write the matrix equation in a form which assumes that the blocks oscillate in a normal mode with angular frequency **ω**.

Stop at this point. Do not solve for the frequencies of the normal modes. Do not pass GO. Do not collect $200.

a) $F_1 = -4kx_1 + k(x_2 - x_1) = -5kx_1 + kx_2$

$F_2 = -k(x_2 - x_1) - kx_2 = kx_1 - 2kx_2$

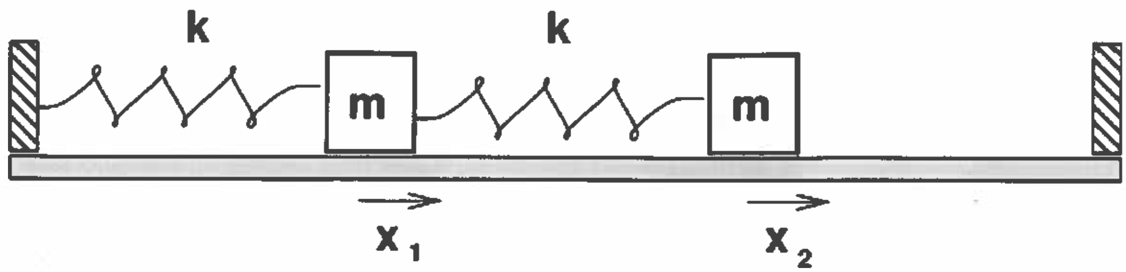
b) $$\begin{bmatrix} -5k & k \\ k & -2k \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3m\, d^2x_1/dt^2 \\ 1m\, d^2x_2/dt^2 \end{pmatrix}$$

$$\left(\frac{k}{m}\right) \begin{bmatrix} -5/3 & 1/3 \\ 1 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} d^2x_1/dt^2 \\ d^2x_2/dt^2 \end{pmatrix}$$

c) $$\left(\frac{k}{m}\right) \begin{bmatrix} -5/3 & 1/3 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\omega^2 \begin{pmatrix} a \\ b \end{pmatrix}$$

or $$\begin{bmatrix} -5/3 & 1/3 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\left(\frac{\omega^2}{k/m}\right) \begin{pmatrix} a \\ b \end{pmatrix}$$

2:00 - 2:07



Problem 2

Two blocks of mass **m** are connected by two identical springs, both with force constant **k**. Fred (correctly) works out the forces on the blocks and puts the result into a matrix equation form like so:

$$\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

where he has defined the quantity λ as

$$\lambda \equiv -\left(\omega^2 \frac{m}{k}\right)$$

a. What is the frequency of one normal mode of this system? (don't bother to solve for both frequencies)
b. What is the eigenvector of that normal mode?

a) $$\begin{bmatrix} -2-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

Find determinant

$$(-2-\lambda)(-1-\lambda) - (1)(1) = 0$$

$$2 + 2\lambda + 1\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 + 3\lambda + 1 = 0$$

$$\lambda = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(1)}}{2}$$

So $-\omega^2 \cdot \frac{m}{k} = -\frac{3}{2} \pm \frac{\sqrt{5}}{2}$

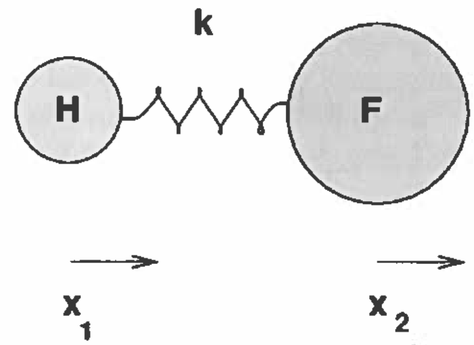
$$\omega^2 = \frac{3}{2} \pm \frac{\sqrt{5}}{2} = \begin{cases} 0.320 \frac{k}{m} \\ 2.618 \frac{k}{m} \end{cases}$$

$$\Rightarrow \boxed{\omega = \begin{cases} 0.618\sqrt{\frac{k}{m}} \\ 1.618\sqrt{\frac{k}{m}} \end{cases}}$$

ω_1: $-2a + b = -0.320$
$\Rightarrow b = 1.618a$ $\begin{pmatrix} 1 \\ 1.618 \end{pmatrix}$

ω_2: $-2a + b = -2.618$
$\Rightarrow b = -0.618a$ $\begin{pmatrix} 1 \\ -0.618 \end{pmatrix}$

2:08 - 2:18



Problem 3

One hydrogen atom of mass **m** and one fluorine atom of mass **19 m** are connected by a molecular bond which acts like an ideal spring of force constant **k**. The molecule is initially at rest, but then absorbs a photon of ordinary yellow light, with wavelength **λ = 550 nm**. As a result, the atoms start to oscillate in the first (lowest energy) normal mode.

a. Write an equation for the frequency of the lowest normal mode of oscillation of the molecule in terms of **k/m**.
b. What is the frequency (Hz) of the photon?
c. What is the angular frequency (rad/s) of the photon?
d. What is the strength of the force constant **k**? (Hint: the mass **m** of a hydrogen atom is given on the front page of this test)

a) $F_1 = +k(x_2 - x_1) = -kx_1 + kx_2 = m_H \frac{d^2x_1}{dt^2}$

$F_2 = -k(x_2 - x_1) = +kx_1 - kx_2 = 19 m_H \frac{d^2x_2}{dt^2}$

$$\rightarrow \begin{bmatrix} -k & +k \\ +k & -k \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} m_H \frac{d^2x_1}{dt^2} \\ 19 m_H \frac{d^2x_2}{dt^2} \end{pmatrix}$$

$$\frac{k}{m_H} \begin{bmatrix} -1 & +1 \\ +\frac{1}{19} & -\frac{1}{19} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{d^2x_1}{dt^2} \\ \frac{d^2x_2}{dt^2} \end{pmatrix}$$

$$\frac{k}{m_H} \begin{bmatrix} -1 & +1 \\ +\frac{1}{19} & -\frac{1}{19} \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\omega^2 \begin{pmatrix} a \\ b \end{pmatrix}$$

Let $\lambda \equiv -\omega^2/k/m$

$$\begin{bmatrix} -1-\lambda & 1 \\ +\frac{1}{19} & -\frac{1}{19}-\lambda \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$\rightarrow (-1-\lambda)(-\frac{1}{19}-\lambda) - (1)(\frac{1}{19}) = 0$

$\lambda^2 + \lambda + \frac{1}{19}\lambda + \cancel{\frac{1}{19}} - \cancel{\frac{1}{19}} = 0$

$\lambda^2 + \frac{20}{19}\lambda = 0$

$\rightarrow \lambda = -\frac{20}{19} = -\omega^2/k/m$

$\rightarrow \omega^2 = \frac{20}{19}\frac{k}{m}$

$\rightarrow \boxed{\omega = \sqrt{\frac{20}{19}\frac{k}{m}}}$

b) $\lambda = 550\text{ nm} = \frac{c}{f}$

$\Rightarrow f = \frac{c}{\lambda} = \boxed{5.45 \times 10^{14}\text{ Hz}}$

c) $\omega = 2\pi f = \boxed{3.42 \times 10^{15}\text{ Hz}}$

d) $\omega^2 = \frac{20}{19}\frac{k}{m} \rightarrow k = \frac{19}{20} m \omega^2$

$= \frac{19}{20}\left(1.67 \times 10^{-27}\text{ kg}\right)\left(3.42 \times 10^{15}\ \frac{1}{s}\right)^2$

$\cong \boxed{18600\ \frac{kg}{s}}$

2:18 ~ 2:25

Problem 4

Your task is to create the world's simplest guitar: just one string, and it must play a single note of frequency **f = 620 Hz.** On the lab bench is a very long (many meters) piece of wire, of radius **r = 0.4 mm**, made out of silver of density **ρ = 11,000 kg/m³**. You can cut the wire to any length you wish, tie it to adjustable rods on the bench, and then stretch it to tension **T = 200 N**.

a. Draw a picture of the wire oscillating in the **n = 3** mode
b. In order to produce the desired note in the **n = 3** mode of oscillation, what length should the wire be?

a)

b) $\lambda = \frac{2}{3}L \qquad f = 620\ \text{Hz} \qquad \omega = 2\pi f = 3{,}896\ \frac{\text{rad}}{s}$

$$\mu = \rho \pi r^2 = \left(11{,}000\ \frac{\text{kg}}{\text{m}^3}\right)\pi\left(0.4\times10^{-3}\ \text{m}\right)^2 = 0.00553\ \frac{\text{kg}}{\text{m}}$$

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{200\ \text{N}}{0.00553\ \text{kg/m}}} = 190\ \frac{\text{m}}{s}$$

But $v = \lambda f = \left(\frac{2}{3}L\right)f$

$$\Rightarrow L = \frac{v}{f}\cdot\frac{3}{2} = \left(\frac{190\ \text{m/s}}{620\ \text{Hz}}\right)\cdot\frac{3}{2} = \boxed{0.46\ \text{m}}$$