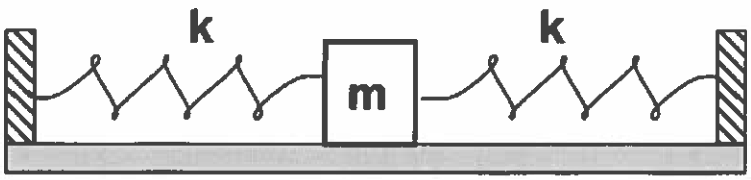
10:28 – 10:35



Problem 1

Joe places a block of mass **m = 4.0 kg** onto a frictionless floor. He attaches it to two identical springs, each of force constant **k = 16 N/m**, so that it sits motionless at equilibrium as shown. Joe pulls the block a distance **d = 8 cm** to the left, holds it motionless, and then at **t = 0**, he releases the block.

a. Just after Joe releases the block, what is the force of the springs on the mass?
b. The block begins to slide back and forth due to the springs. What is the period of motion?
c. What is the maximum speed of the block?
d. Write an equation for the position of the block as a function of time. Provide numerical values, with units, for all coefficients and parameters in the equation.

a) $\vec{F} = +kx + kx = 2kx = 2\left(16\,\frac{N}{m}\right)(0.08\,m) = \boxed{2.56\,N \text{ to right}}$

b) $\omega = \sqrt{\frac{2k}{m}} = \sqrt{\frac{2\cdot 16\,N/m}{4.0\,kg}} = 2.83\,\frac{rad}{s}$

$\boxed{P = \frac{2\pi}{\omega} = 2.22\,s}$

c) $\max PE = \frac{1}{2}kx_{max}^2 + \frac{1}{2}kx_{max}^2 = kx_{max}^2 = \left(16\,\frac{N}{m}\right)(0.08\,m)^2 = 0.102\,J$

$\max PE = \max KE = \frac{1}{2}mv_{max}^2 = 0.102\,J$

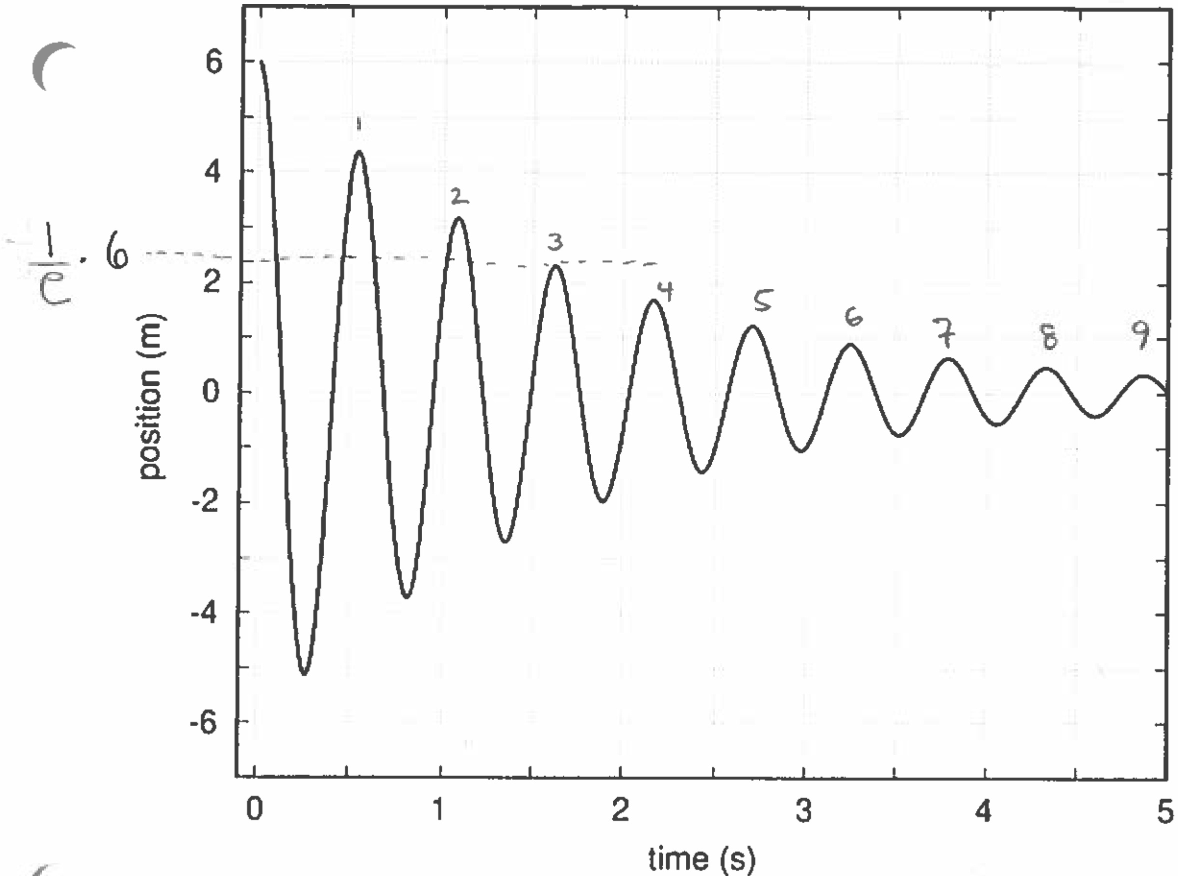
$\Rightarrow v_{max} = \sqrt{\frac{2\cdot(0.102\,J)}{4.0\,kg}} = \boxed{0.227\,\frac{m}{s}}$

Check: should also be ωA
$= \left(2.83\,\frac{rad}{s}\right)(0.08\,m)$
$= 0.226\,\frac{m}{s}$ ✓

d) $x(t) = (0.08\,m)\cos\left(2.83\,\frac{rad}{s}\cdot t + \pi\,rad\right)$

$= -(0.08\,m)\cos\left(2.83\,\frac{rad}{s}\cdot t\right)$

10:35 – 10:49



Problem 2

The graph above shows the position of a ball attached to a spring as a function of time.

a. Write the GENERAL equation for the position of the ball as a function of time in this situation. Use symbols, without numbers.
b. Write the SPECIFIC equation for the position of the ball as a function of time, using information shown in the graph. Include numbers with units for all parameters and constants.
c. The force constant of the spring is **k = 27 N/m**. Estimate the mass of the ball; your answer doesn't need to be exact.
d. Is this a high-Q or low-Q system? Justify your answer.

a) $x(t) = A\cos(\omega t + \phi)e^{-t/\tau}$

b) Easy version: $A = 6\text{ m}$, $\phi = 0\text{ rad}$. $\frac{1}{e}\cdot 6\text{m} = 2.20\text{ m}$ at $t \approx 1.6\text{s}$ $\Rightarrow \tau = 1.6\text{s}$

9 cycles takes $\approx 4.8\text{ s} \Rightarrow T = \frac{4.8\text{ s}}{9\text{ cycle}} = 0.53\text{ s}$

$\Rightarrow \omega = \frac{2\pi\text{ rad}}{0.53\text{ s}} = 11.8\text{ rad/s}$

$\Rightarrow \boxed{x(t) = (6\text{ m})\cos\left(11.8\tfrac{\text{rad}}{\text{s}}\cdot t + 0\text{ rad}\right)e^{-t/1.6\text{s}}}$

this gets full credit

b) Harder version: assume $V(t=0) = 0\ ^{m}/_{s}$

$$\Rightarrow v(t) = -\omega A \sin(\omega t + \phi) e^{-t/\tau} - \frac{1}{\tau} A \cos(\omega t + \phi)$$

$$\Rightarrow v(0) = -\omega A \sin\phi - \frac{1}{\tau} A \cos\phi = 0$$

$$\Rightarrow \frac{\sin\phi}{\cos\phi} = -\frac{1}{\omega\tau} \qquad \tan\phi = -\frac{1}{(11.8\frac{rad}{s})(1.6s)}$$

$$\Rightarrow \phi = \tan^{-1}(-0.053) \cong -0.053\ rad \approx 3°$$

$$x(t=0) = A\cos\phi = 6m$$

$$\Rightarrow A = \frac{6m}{\cos\phi} \cong 6.01m$$

$$\boxed{x(t) = (6.01m)\cos\left(11.8\tfrac{rad}{s}t - 0.053\ rad\right) e^{-t/1.6s}}$$

+1 bonus

c) $K = 27\frac{N}{m}$ $\qquad \omega_0 = \sqrt{\frac{K}{m}}$ and in this case $\omega_0 \approx \omega$

So $\omega \approx \sqrt{\frac{K}{m}} \Rightarrow m \approx \frac{K}{\omega^2} = \boxed{0.19\ kg}$

Not exact but we don't know b

d) $Q = \#$ cycles for amplitude to fall to $\frac{1}{e} \cdot \pi \approx 3$ cycles $\cdot\ \pi$

$Q \cong 10$ $\qquad$ so $\boxed{\text{low } Q}$

10:49 – 11:02

Problem 3

Joe attaches a wooden plank of **m = 4.000 kg** to a spring of force constant **k = 15.00 kg**. He allows it to come to rest, then stretches it exactly **A_0 = 100.0 cm** from its equilibrium position. At time **t = 0 s**, he releases it, and it starts to oscillate up and down. However, air resistance quickly damps the motion, and at time **t = 10.00 s**, the amplitude of motion has shrunk to just **A = 13.53 cm**.

a. What is the natural frequency of this system?
b. What is the resistive coefficient **b** of this system?
c. During those ten seconds, exactly how many cycles of oscillation did the plank make? Express your answer to three decimal places.

Fred now attaches a circular motor to the spring, which exerts a small force in a periodic manner: **F = 5.000 N * sin(ω_d t)**

d. If Fred sets the driving frequency to **ω_d = 1.000 rad/s**, what will the be the steady-state amplitude of the plank's motion?

a) $\omega_0 = \sqrt{\frac{k}{m}} = \boxed{1.936 \frac{rad}{s}}$

b) Note that in $t = 10$ s, amplitude of motion drops to $\frac{13.53}{100} = 0.135$ which is $\frac{1}{e} \cdot \frac{1}{e}$ of original. So that's two time constants

see note below

$2\tau = 10.00\text{ s} \Rightarrow \tau = 5.00\text{ s}$

but $\tau = \frac{2m}{b} \Rightarrow b = \frac{2m}{\tau} = \frac{2(4.000\text{kg})}{5.000\text{s}} = \boxed{1.600 \frac{kg}{s}}$

c) $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{3.71 \frac{1}{s^2}} = 1.926 \frac{rad}{s} \Rightarrow P = \frac{2\pi}{\omega} = 3.262$

So in 10 sec, # cycles $= \frac{10\text{ s}}{3.262\text{ s}} = \boxed{3.066 \text{ cycles}}$

get 3.082 cycl if use ω_0

d) $A = \frac{F_{max}/m}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \frac{b^2\omega^2}{m^2}}} = \frac{5N / 4\text{ kg}}{\sqrt{\left(1.936\frac{rad}{s}^2 - 1\frac{rad}{s}^2\right)^2 + \frac{(1.6\frac{kg}{s})^2(1\frac{rad}{s})^2}{(4kg)^2}}}$

$\boxed{A = 0.45\text{ m}}$

Note $A = 2.59$ m if use $\omega_d = 1.926$ rad/s

b) Another way to find time constant τ

$$A_0(t=0) = 100\text{ cm} \quad = A_0 e^{-0/\tau}$$

$$A_1(t=10) = 13.53\text{ cm} \quad = A_0 e^{-10/\tau}$$

Divide the two amplitudes

$$\frac{A_1}{A_0} = \frac{13.53\text{ cm}}{100\text{ cm}} = \frac{\cancel{A_0} e^{-10/\tau}}{\cancel{A_0} \underbrace{e^{-0/\tau}}_{=1}} = e^{-10/\tau}$$

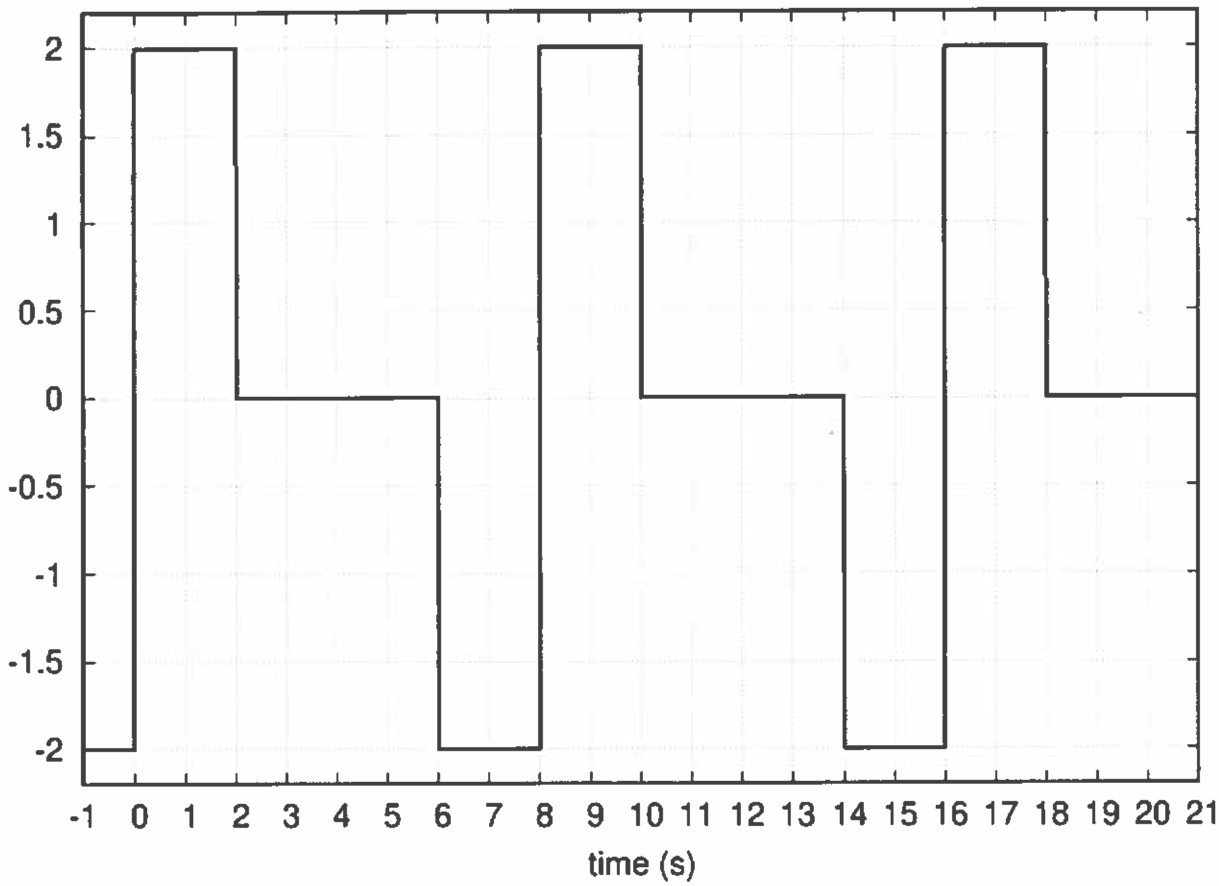
$$\Rightarrow \quad e^{-10/\tau} = 0.1353$$

$$\ln\left(e^{-10/\tau}\right) = \ln(0.1353)$$

$$-10/\tau = \ln(0.1353)$$

$$\tau = -\frac{10}{\ln(0.1353)} = \frac{-10}{-2} = \boxed{5}$$

11:02 - 11:07



Problem 4

The graph above shows a periodic function of time.

a. What is the period of this function?
b. Write an equation which could be used to compute the Fourier coefficient $\mathbf{A_0}$. Do not evaluate the equation yet.
c. Evaluate the equation to compute the value of $\mathbf{A_0}$.
d. Write an equation which could be used to compute the Fourier coefficient $\mathbf{A_1}$. Do not evaluate the equation yet.
e. Evaluate the equation to compute the value of $\mathbf{A_1}$.

2

a) $P = 8s$

2 b) $A_0 = \frac{1}{P}\int_0^P f(t)\,dt$

2 c) $A_0 = \frac{1}{8}\int_0^2 2\,dt + \frac{1}{8}\int_2^6 0\,dt + \frac{1}{8}\int_6^8 -2\,dt = \frac{1}{2} + 0 - \frac{1}{2} = 0$

2 d) $A_1 = \frac{2}{P}\int_0^P f(t)\cos\left(\frac{2\pi}{P}t\right)dt$

e) $A_1 = \frac{2}{8}\int_0^2 2\cdot\cos\left(\frac{\pi}{4}t\right)dt + 0 + \frac{2}{8}\int_6^8 (-2)\cos\left(\frac{\pi}{4}t\right)dt$

2

$= \frac{1}{2}\cdot\frac{4}{\pi}\left[\sin\left(\frac{2\pi}{4}\right) - \sin(0)\right] + 0 - \frac{1}{2}\cdot\frac{4}{\pi}\left[\sin(2\pi) - \sin\left(\frac{3\pi}{2}\right)\right]$

$$A_1 = \frac{2}{\pi}\left[1-0\right] - \frac{2}{\pi}\left[0-(-1)\right]$$

$$= \frac{2}{\pi} - \frac{2}{\pi} = 0$$