$$y_1(x,t) \quad = \quad A\sin(kx - \omega t)$$

$$y_2(x,t) \quad = \quad A\sin(kx + \omega t)$$

$$y_1 + y_2 \quad = \quad A\left[\sin(kx - \omega t) \ + \ \sin(kx + \omega t)\right]$$

$$\sin(\alpha)\cos(\beta) \quad = \quad \frac{1}{2}\left[\sin(\alpha - \beta) \ + \ \sin(\alpha + \beta)\right]$$

$$y_1 + y_2 \quad = \quad 2A\left[\sin(kx)\ \cos(\omega t)\right]$$

$$y_1(x,t) \quad = \quad A\sin(k_1 x \ - \ \omega_1 t)$$

$$y_2(x,t) \quad = \quad A\sin(k_2 x \ - \ \omega_2 t)$$

$$k_1 \quad \simeq \quad k_2 \qquad \Rightarrow \quad k_2 \quad = \quad k_1 \ + \ \Delta k$$

$$\omega_1 \quad \simeq \quad \omega_2 \qquad \Rightarrow \quad \omega_2 \quad = \quad \omega_1 \ + \ \Delta\omega$$

$$v_1 \quad = \quad \frac{\omega_1}{k_1} \quad = \quad v_2 \quad = \quad \frac{\omega_2}{k_2}$$

$$y_1 + y_2 \quad = \quad A\sin(k_1 x \ - \ \omega_1 t) \ + \ A\sin(k_2 x \ - \ \omega_2 t)$$

$$\sin(\alpha) \ + \ \sin(\beta) \quad = \quad 2\ \sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\alpha \quad = \quad k_1 x - \omega_1 t \quad , \qquad \beta \quad = \quad k_2 x - \omega_2 t$$

$$\begin{aligned}
y_1 + y_2 \quad &= \quad 2A\ \sin\left(\frac{k_1 x - \omega_1 t + k_2 x - \omega_2 t}{2}\right)\ \cos\left(\frac{k_1 x - \omega_1 t - k_2 x + \omega_2 t}{2}\right) \\
&= \quad 2A\ \sin\left((\frac{k_1 + k_2}{2})x - (\frac{\omega_1 + \omega_2}{2})t\right)\ \cos\left((\frac{k_1 - k_2}{2})x - (\frac{\omega_1 - \omega_2}{2})t\right) \\
&= \quad 2A\ \sin\left(k_{\mathrm{avg}}\, x \ - \ \omega_{\mathrm{avg}}\, t\right)\ \cos\left(\frac{\Delta k}{2}\, x \ - \ \frac{\Delta\omega}{2}\, t\right)
\end{aligned}$$

$$y_1 + y_2 = 2A \sin\left(k_{\mathrm{avg}}\, x - \omega_{\mathrm{avg}}\, t\right) \cos\left(\frac{\Delta k}{2}\, x - \frac{\Delta\omega}{2}\, t\right)$$

$$v_p = \frac{\omega_{\mathrm{avg}}}{k_{\mathrm{avg}}}$$

$$v_g = \frac{\Delta\omega/2}{\Delta k/2} = \frac{\Delta\omega}{\Delta k}$$

$$v_g = \frac{d\omega}{dk}$$

$$\omega_1 = \omega \qquad \text{(no change)}$$

$$\lambda_1 = \frac{\lambda}{n}$$

$$k_1 = \frac{2\pi}{\lambda_1} = \frac{2\pi n}{\lambda} = nk$$

$$v_p = \frac{\omega_1}{k_1} = \frac{\omega}{nk} = \frac{1}{n}\,\frac{\omega}{k}$$

$$v_p = \frac{c}{n}$$

$$v_p = \frac{\omega}{k} \quad \Rightarrow \quad \omega = k\, v_p$$

$$\omega = \frac{kc}{n}$$

$$\frac{d\omega}{dk} = \frac{d}{dk}\left(\frac{kc}{n}\right)$$

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}\left(\frac{kc}{n}\right)$$

$$\frac{d\omega}{dk} = \frac{c}{n} + kc\,\frac{d}{dk}\left(\frac{1}{n}\right) = \frac{c}{n} + kc\left(-\frac{1}{n^2}\,\frac{dn}{dk}\right)$$

$$\frac{d\omega}{dk} = \frac{c}{n} - \frac{kc}{n^2}\,\frac{dn}{dk}$$

$$v_g = \frac{d\omega}{dk} = \frac{c}{n} - \frac{c}{n}\left(\frac{k}{n}\,\frac{dn}{dk}\right) = \frac{c}{n}\left(1 - \frac{k}{n}\,\frac{dn}{dk}\right)$$

$$v_g = v_p\left(1 - \frac{k}{n}\frac{dn}{dk}\right)$$

$$\frac{dn}{d\lambda} > 0 \quad \text{or} \quad \frac{dn}{dk} < 0$$