$$\begin{aligned} \text{x} &= \text{location on the rod} \\ \text{z} &= \text{displacement from original location} \end{aligned}$$

$$dm = \rho V = \rho A\, dx$$

$$\text{stress} = \frac{\text{force}}{\text{area}} = \frac{F_2 - F_1}{A}$$

$$\text{strain} = \frac{\text{change in length}}{\text{original length}} = \frac{\Delta z}{\Delta x}$$

$$\text{stress} = Y \text{ strain}$$

$$\left(\frac{F_2 - F_1}{A}\right) = Y \left(\frac{\Delta z}{\Delta x}\right)$$

$$F_2 - F_1 = (\rho\ A\ \Delta x)\ \frac{\partial^2 z}{\partial t^2}$$

$$\frac{F_2 - F_1}{A} = (\rho\ \Delta x)\ \frac{\partial^2 z}{\partial t^2}$$

$$\left(\frac{F_2 - F_1}{A}\right) = Y \left(\frac{\Delta z}{\Delta x}\right)$$

$$\Delta z = \frac{\partial z}{\partial x}\ \Delta x$$

$$\frac{\Delta z}{\Delta x} \simeq \frac{\partial}{\partial x}(\Delta z) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\ \Delta x\right) = \frac{\partial^2 z}{\partial x^2}\ \Delta x$$

$$\left(\frac{F_2 - F_1}{A}\right) = Y \left(\frac{\Delta z}{\Delta x}\right) \quad \Rightarrow \quad (\rho\ \ \Delta x)\,\frac{\partial^2 z}{\partial t^2} = Y\ \frac{\partial^2 z}{\partial x^2}\ \Delta x$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{\rho}{Y}\right)\frac{\partial^2 z}{\partial t^2}$$

$$\frac{1}{v^2} = \frac{\rho}{Y} \quad \Rightarrow \quad v = \sqrt{\frac{Y}{\rho}}$$

$$\lambda_n = \frac{4L}{2n-1}$$

$$f_n = \frac{v}{\lambda_n} = \sqrt{\frac{Y}{\rho}}\ \left(\frac{2n-1}{4L}\right)$$