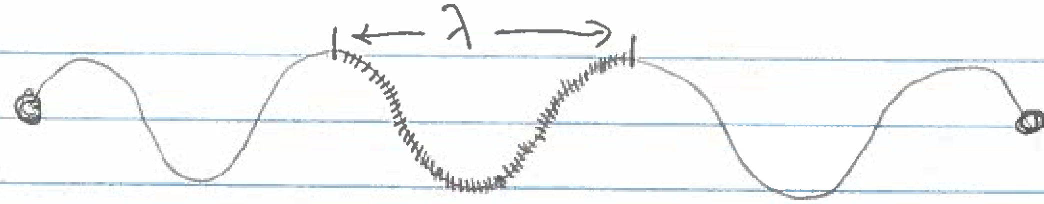


A long rope stretched between two posts, under tension $T = 15\ N$, is plucked. It oscillates in one of the normal modes, and a wave travels down the rope

$$y = A \sin(kx - \omega t)$$

where

$$A = 0.05\ m$$

$$\lambda = 1.8\ m \qquad \text{so} \qquad k = \frac{2\pi}{\lambda} = 3.49\ \frac{rad}{m}$$

The marked section of rope is exactly 1 wavelength long and has a mass

$$m = 0.051\ kg$$

$$\rightarrow \qquad \mu = \frac{m}{\lambda} = \frac{0.051\ kg}{1.8\ m} = 0.0283\ \frac{kg}{m}$$

At time $t=0$, Fred takes a snapshot of this section of the rope. Thus,

$$y(x, t=0) = A \sin(kx - \omega \cdot 0) = A \sin(kx)$$

$$\frac{\partial y}{\partial t}(x, t=0) = -\omega A \cos(kx - \omega \cdot 0) = -\omega A \cos(kx)$$

The kinetic energy of the section of rope can be computed as the integral of many little pieces of rope:

| | | |
|---|---|---|
| little piece | length | $dx$ |
| " " | mass | $dm = \mu dx$ |
| " " | y-velocity | $v_y = \partial y/\partial t$ |

So

$$\text{little KE} \quad dKE = \frac{1}{2} dm\, v_y^2 = \frac{1}{2}(\mu dx)\left(-\omega A \cos(kx)\right)^2$$

$$= \frac{1}{2}\mu dx\, \omega^2 A^2 \cos^2(kx)$$

So the KE of the section will be

$$KE = \int_{1 \text{ wavelength}} dKE = \int_{x=0}^{x=\lambda} \frac{1}{2}\mu\omega^2 A^2 \cos^2(kx)\, dx$$

$$= \frac{1}{2}\mu\omega^2 A^2 \int_0^\lambda \cos^2(kx)\, dx$$

But

$$k = \frac{2\pi}{\lambda}$$

$$KE = \frac{1}{2}\mu\omega^2 A^2 \int_0^\lambda \cos^2\left(\frac{2\pi}{\lambda}x\right)$$

Now, the integral of $\cos^2(ax) = \frac{1}{2}x + \frac{1}{4a}\sin(2ax)$, so

$$KE = \frac{1}{2}\mu\omega^2A^2\left[\frac{1}{2}x - \frac{1}{4\cdot\frac{2\pi}{\lambda}}\sin\left(2\frac{2\pi}{\lambda}x\right)\right|_0^\lambda$$

$$= \frac{1}{2}\mu\omega^2A^2\left[\frac{1}{2}\lambda - \frac{\lambda}{8\pi}\cancel{\sin\left(\frac{4\pi\lambda}{\lambda}\right)} - \left\{\cancel{\frac{1}{2}\cdot 0} - \cancel{\frac{\lambda}{8\pi}\sin(0)}\right\}\right]$$

$$= \frac{1}{2}\mu\omega^2A^2\left(\frac{1}{2}\lambda\right)$$

$$= \frac{1}{4}\mu\omega^2A^2\lambda$$

What is $\omega$? We know

$$v = \sqrt{\frac{T}{\mu}} = \frac{\omega}{k} = \frac{\omega}{2\pi/\lambda} = \frac{\omega\lambda}{2\pi}$$

So

$$\omega = \frac{2\pi}{\lambda}\sqrt{\frac{T}{\mu}} = \left(\frac{2\pi}{1.8\,m}\right)\sqrt{\frac{15\,N}{0.0283\,kg/m}}$$

$$= 80.3\,\frac{rad}{s}$$

So we can compute

$$KE = \frac{1}{4}\left(0.0283\,\frac{kg}{m}\right)\left(80.3\,\frac{rad}{s}\right)^2(0.05\,m)^2(1.8\,m)$$

$$\boxed{KE = 0.205\,J}$$

What is the potential energy of this section of rope?
We know the potential energy of a tiny piece of rope will be

$$dPE = \frac{1}{2} T \left(\frac{\partial y}{\partial x}\right)^2 dx$$

$$= \frac{1}{2} T k^2 A^2 \cos^2(kx - \omega t) \, dx$$

but $t=0$, so

$$= \frac{1}{2} T k^2 A^2 \cos^2(kx) \, dx$$

Once again, we must integrate over the section of rope

$$PE = \int_{\text{one wavelength}} dPE = \int_0^\lambda \frac{1}{2} T k^2 A^2 \cos^2(kx) \, dx$$

$$= \frac{1}{2} T k^2 A^2 \underbrace{\int_0^\lambda \cos^2(kx) \, dx}_{= \frac{1}{2}\lambda}$$

from our previous integration

$$= \frac{1}{4} T k^2 A^2 \lambda$$

$$= \frac{1}{4} (15 \, N) \left(3.49 \, \frac{rad}{m}\right)^2 (0.05 \, m)^2 (1.8 \, m)$$

$$\boxed{PE = 0.205 \, J}$$

same as KE!