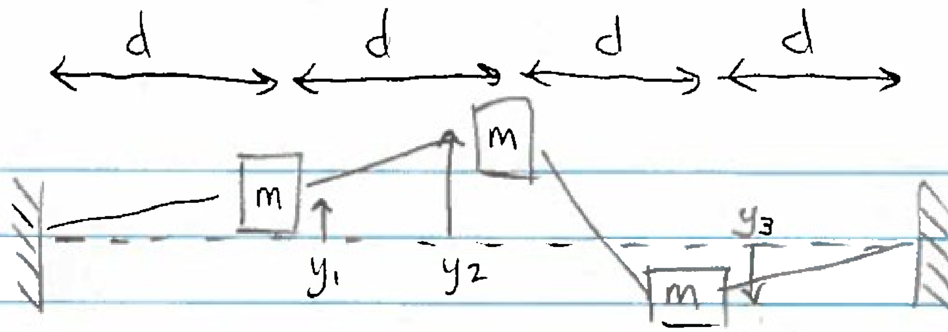
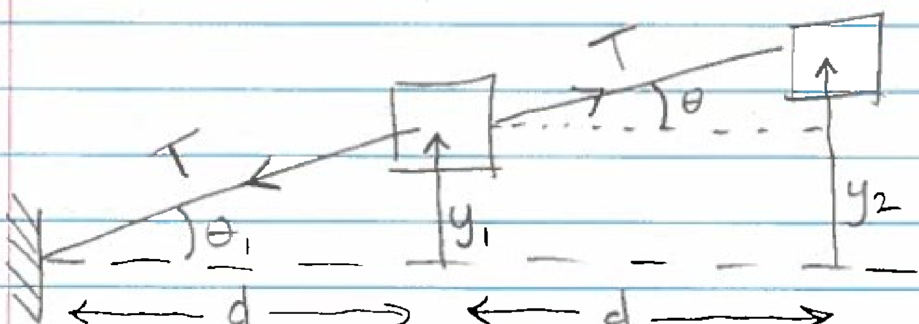


Blocks of mass m are connected by string with uniform tension T. The horizontal spacing of the blocks is d. What are the frequencies of the normal modes of this system?



We must begin by computing the forces on each block. Consider block 1. The vertical forces on it are

$$F_1 = -T\sin\theta_1 + T\sin\theta_2$$

but for small displacements, $y << d$, we can write

$$\sin\theta_1 = \frac{y_1}{\sqrt{d^2+y_1^2}} \approx \frac{y_1}{d}$$

$$\sin\theta_2 = \frac{y_2-y_1}{\sqrt{d^2+(y_2-y_1)^2}} \approx \frac{y_2-y_1}{d}$$

and so the forces on block 1 can be written

$$F_1 = -T\frac{y}{d} + T\frac{y_2 - y_1}{d} = -2\frac{T}{d}y_1 + \frac{T}{d}y_2$$

In a similar manner, one can compute

$$F_2 = -T\frac{y_2 - y_1}{d} + T\frac{y_3 - y_2}{d} = \frac{T}{d}y_1 - 2\frac{T}{d}y_2 + \frac{T}{d}y_3$$

$$F_3 = -T\frac{y_3 - y_2}{d} - T\frac{y_3}{d} = \frac{T}{d}y_2 - 2\frac{T}{d}y_3$$

If we write each force as mass time acceleration, we find

$$-2\frac{T}{d}y_1 + \frac{T}{d}y_2 = m\frac{d^2y_1}{dt^2}$$

$$1\frac{T}{d}y_1 - 2\frac{T}{d}y_2 + 1\frac{T}{d}y_3 = m\frac{d^2y_2}{dt^2}$$

$$1\frac{T}{d}y_2 - 2\frac{T}{d}y_3 = m\frac{d^2y_3}{dt^2}$$

Another way to write this is as a matrix equation

$$\left(\frac{T}{d}\right)\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = m\begin{pmatrix} \frac{d^2y_1}{dt^2} \\ \frac{d^2y_2}{dt^2} \\ \frac{d^2y_3}{dt^2} \end{pmatrix}$$

If we assume that there is some linear combination of the positions of the blocks

$$s = a y_1 + b y_2 + c y_3$$

which yields a <u>normal mode</u> (all blocks move with the same frequency ω), then we can write

$$\left(\frac{T}{md}\right)\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\omega^2 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The eigenvalues of this matrix can be used to compute the frequencies of the normal modes

eigenvalue -2 $\Rightarrow$ $-\omega_1^2 = -2\frac{T}{md}$

$$\Rightarrow \omega_1 = \sqrt{2\frac{T}{md}}$$

eigenvalue $\sqrt{2}-2$ $\Rightarrow$ $-\omega_2^2 = (\sqrt{2}-2)\frac{T}{md}$

$$\Rightarrow \omega_2 = \sqrt{(2-\sqrt{2})\frac{T}{md}}$$

eigenvalue $-2-\sqrt{2}$ $\Rightarrow$ $-\omega_3^2 = (-2-\sqrt{2})\frac{T}{md}$

$$\Rightarrow \omega_3 = \sqrt{(2+\sqrt{2})\frac{T}{md}}$$

And so all three frequencies of the normal modes have a factor of

$$\sqrt{\frac{T}{md}}$$

lowest freq: $\sqrt{2-\sqrt{2}}\ \sqrt{\frac{T}{md}} = 0.77\ \sqrt{\frac{T}{md}}$

middle freq: $\sqrt{2}\ \sqrt{\frac{T}{md}} = 1.41\ \sqrt{\frac{T}{md}}$

highest freq: $\sqrt{2+\sqrt{2}}\ \sqrt{\frac{T}{md}} = 1.85\ \sqrt{\frac{T}{md}}$