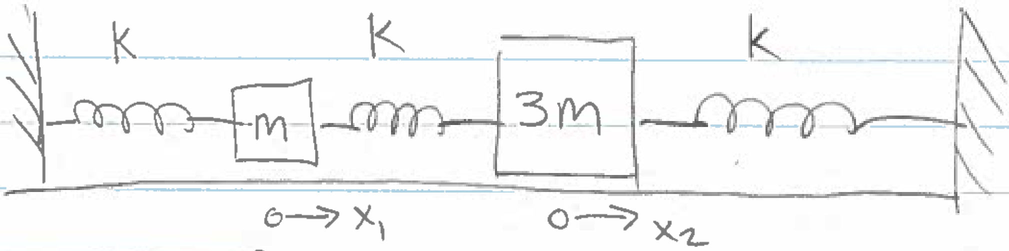

A system of two blocks,

small $m = 6\,kg$

big $3m = 18\,kg$

is connected by identical springs

$k = 10\ N/m$

Fred sets initial conditions so

$x_1 = 0\ m$

$x_2 = +L\ m$

$v_1 = 0\ m/s$

$v_2 = 0\ m/s$

When he releases the blocks, they begin to slide back and forth. What are the angular frequencies of the normal modes?

$$F_1 = m\frac{d^2x_1}{dt^2} = -kx_1 + (x_2 - x_1)k = -2kx_1 + kx_2$$

$$F_2 = 3m\frac{d^2x_2}{dt^2} = -kx_2 - (x_2 - x_1)k = +kx_1 - 2kx_2$$

Re-arrange into matrix form

$$\begin{pmatrix} -\frac{2k}{m} & +\frac{k}{m} \\ +\frac{k}{3m} & -\frac{2k}{3m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{d^2x_1}{dt^2} \\ \frac{d^2x_2}{dt^2} \end{pmatrix}$$

To find eigenvalues, we assume some linear combination

$$s = a x_1 + b x_2$$

will yield SHM. Thus,

$$\begin{pmatrix} -2\frac{k}{m} & +\frac{k}{m} \\ +\frac{k}{3m} & -\frac{2k}{3m} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\omega^2 \begin{pmatrix} a \\ b \end{pmatrix}$$

(1) $$-2\frac{k}{m}a + \frac{k}{m}b + \omega^2 a = 0$$

(2) $$+\frac{1}{3}\frac{k}{m}a - \frac{2}{3}\frac{k}{m}b + \omega^2 b = 0$$

From (1)

$$b = \left(\frac{2\frac{k}{m} - \omega^2}{k/m}\right) a$$

Insert into (2)

$$\frac{1}{3}\frac{k}{m}a - \frac{2}{3}\frac{k}{m}\left(\frac{2\frac{k}{m} - \omega^2}{k/m}\right)a + \omega^2\left(\frac{2\frac{k}{m} - \omega^2}{k/m}\right)a = 0$$

$$\frac{1}{3}\left(\frac{k}{m}\right)^2 a - \frac{4}{3}\left(\frac{k}{m}\right)^2 a + \frac{2}{3}\frac{k}{m}\omega^2 a + 2\frac{k}{m}\omega^2 a - \omega^4 a = 0$$

$$-1\left(\frac{k}{m}\right)^2 + \frac{8}{3}\frac{k}{m}\omega^2 - \omega^4 = 0$$

Now, let $\lambda \equiv \omega^2$.

We end up with quadratic equation for λ.

$$\lambda^2 - \frac{8}{3}\frac{k}{m}\lambda + \left(\frac{k}{m}\right)^2 = 0$$

The solution is

$$\lambda = \frac{\frac{8}{3}\frac{k}{m} \pm \sqrt{\frac{64}{9}\left(\frac{k}{m}\right)^2 - 4\left(\frac{k}{m}\right)^2}}{2}$$

$$= \frac{\frac{8}{3}\frac{k}{m} \pm \sqrt{\frac{64}{9}\left(\frac{k}{m}\right)^2 - \frac{36}{9}\left(\frac{k}{m}\right)^2}}{2}$$

$$= \frac{\frac{8}{3}\frac{k}{m} \pm \sqrt{\frac{28}{9}\left(\frac{k}{m}\right)^2}}{2}$$

$$= \frac{\frac{8}{3}\frac{k}{m} \pm \sqrt{\frac{4}{9}\left(\frac{k}{m}\right)^2 \cdot 7}}{2}$$

$$= \frac{\frac{8}{3}\frac{k}{m} \pm \frac{2}{3}\frac{k}{m}\sqrt{7}}{2}$$

$$= \frac{\left(\frac{8}{3} \pm \frac{2\sqrt{7}}{3}\right)\frac{k}{m}}{2}$$

$$\lambda = \left(\frac{4}{3} \pm \frac{\sqrt{7}}{3}\right)\frac{k}{m} = \frac{4 \pm \sqrt{7}}{3}\frac{k}{m}$$

And so the frequencies of the normal modes are

$$\omega^2 = \frac{4 \pm \sqrt{7}}{3} \left(\frac{k}{m}\right)$$

$$\rightarrow \boxed{\begin{aligned} \omega_1 &= \sqrt{\frac{4-\sqrt{7}}{3}\left(\frac{k}{m}\right)} = 0.4514 \sqrt{\frac{k}{m}} \ \frac{\text{rad}}{\text{s}} \\ \omega_2 &= \sqrt{\frac{4+\sqrt{7}}{3}\left(\frac{k}{m}\right)} = 2.215 \sqrt{\frac{k}{m}} \ \frac{\text{rad}}{\text{s}} \end{aligned}}$$

To find the position of left-hand block at some particular time, we need to

1) find the eigenvectors corresponding to each frequency

2) use initial conditions to determine constants of integration in the normal modes (A, ϕ)

3) express position $x_1(t)$ in terms of the normal modes $s_1(t)$ and $s_2(t)$

Step 1): Find eigenvector for each frequency.

Use ω_1

$$-2\frac{k}{m} + \frac{k}{m} b + \underbrace{\left(\frac{4-\sqrt{7}}{3}\right)\left(\frac{k}{m}\right)}_{\omega_1^2} a = 0 \rightarrow$$

$$\frac{k}{m} b = \left(2 - \left(\frac{4-\sqrt{7}}{3}\right)\right)\left(\frac{k}{m}\right) a$$

$$\rightarrow \quad b = \left\{2 - \frac{4-\sqrt{7}}{3}\right\} a$$

Define

$$\alpha \equiv 2 - \left(\frac{4-\sqrt{7}}{3}\right) = 1.549$$

Then

$$b = \alpha a$$

$$\rightarrow \boxed{S_1 = 1 x_1 + \alpha x_2} = x_1 + 1.549 x_2$$

Now, for eigenvalue $\omega_2 = \sqrt{\frac{4+\sqrt{7}}{3}\left(\frac{k}{m}\right)}$

$$-2\frac{k}{m} a + \frac{k}{m} b + \left(\frac{4+\sqrt{7}}{3}\right)\left(\frac{k}{m}\right) a = 0$$

$$\frac{k}{m} b = \left\{2 - \frac{4+\sqrt{7}}{3}\right\}\left(\frac{k}{m}\right) a$$

Define

$$\beta \equiv 2 - \left(\frac{4+\sqrt{7}}{3}\right) = -0.2153$$

Then

$$b = \beta a$$

$$\boxed{S_2 = 1 x_1 + \beta x_2} = x_1 - 0.2153 x_2$$

Step 2) use initial conditions to find A_1, ϕ_1, A_2, ϕ_2

$$s_1(t) = A_1 \cos(\omega_1 t + \phi_1) = x_1 + \alpha x_2$$

$$s_2(t) = A_2 \cos(\omega_2 t + \phi_2) = x_1 + \beta x_2$$

$$\frac{ds_1}{dt}(t) = -\omega_1 A_1 \sin(\omega_1 t + \phi_1) = v_1 + \alpha v_2$$

$$\frac{ds_2}{dt}(t) = -\omega_2 A_2 \sin(\omega_2 t + \phi_2) = v_1 + \beta v_2$$

Now, at $t=0$, $v_1 = 0$ and $v_2 = 0$, which means that (thank goodness)

$$\phi_1 = 0$$
$$\phi_2 = 0$$

Also at $t=0$,

$$x_1 = 0$$
$$x_2 = +L$$

So

$$s_1(t=0) = A_1 \cos(0t + 0) = A_1 = 0 + \alpha L$$

$$\rightarrow A_1 = \alpha L$$

$$s_2(t=0) = A_2 \cos(0t + 0) = A_2 = 0 + \beta L$$

$$\rightarrow A_2 = \beta L$$

And so we have

$$s_1(t) = \alpha L \cos(\omega_1 t) = 1.549 L \cos(\omega_1 t)$$

$$s_2(t) = \beta L \cos(\omega_2 t) = -0.2153 L \cos(\omega_2 t)$$

Step 3) Express $x_1(t)$ in terms of $s_1(t)$ and $s_2(t)$

$$s_1 = x_1 + \alpha x_2$$
$$s_2 = x_1 + \beta x_2$$

If we multiply first equation, both sides, by $-\beta/\alpha$

$$-\beta/\alpha \, s_1 = -\beta/\alpha \, x_1 + (-\beta/\alpha)\alpha x_2$$

$$-\beta/\alpha \, s_1 = -\beta/\alpha \, x_1 - \beta x_2$$

Now add $\quad + \; s_2 = x_1 + \beta x_2$

$$s_2 - \frac{\beta}{\alpha} s_1 = x_1 - \frac{\beta}{\alpha} x_1 + 0$$

We can solve for x_1 in terms of s_1 and s_2

$$\left(1 - \frac{\beta}{\alpha}\right) x_1 = s_2 - \beta/\alpha \, s_2$$

$$\rightarrow \boxed{x_1 = \frac{s_2 - \beta/\alpha \, s_1}{1 - \beta/\alpha}}$$

And so finally we can write $x_1(t)$

$$x_1(t) = \frac{\beta L \cos(\omega_2 t) - (\beta/\alpha)\, \alpha L \cos(\omega_1 t)}{1 - \beta/\alpha}$$

$$= \left[\frac{L}{1-\beta/\alpha}\right]\left(\beta \cos(\omega_2 t) - \beta \cos(\omega_1 t)\right)$$

Put in numbers for $\alpha = 1.549$ $\beta = -0.2153$

$$\boxed{x_1(t) = \left[\frac{L}{1.139}\right]\left(-0.2153 \cos(\omega_2 t) + 0.2153 \cos(\omega_1 t)\right)}$$

Note that $x_1(t=0) = 0$, matching given initial condition.

Plug in given time, and compute $x_1(t)$.