Captain Smith fires shells at $V_0 = 800\ \frac{m}{s}$ at an angle $\theta = 30°$ above horizontal. How far do shells travel?

Idea: since path of shells is symmetric, find time to reach the peak, t_{peak}; then multiply by 2 to get total time in air.

$$V_x = V_0 \cos\theta$$
$$V_y = V_0 \sin\theta - gt$$

When at peak, $V_y = 0$.

$$V_y = 0 = V_0 \sin\theta - g t_{peak}$$

$$V_0 \sin\theta = g t_{peak}$$

$$t_{peak} = \frac{V_0 \sin\theta}{g} = \frac{800\ \frac{m}{s} \cdot \sin 30°}{9.8\ m/s^2}$$

$$= 40.8\ s$$

So total time in air

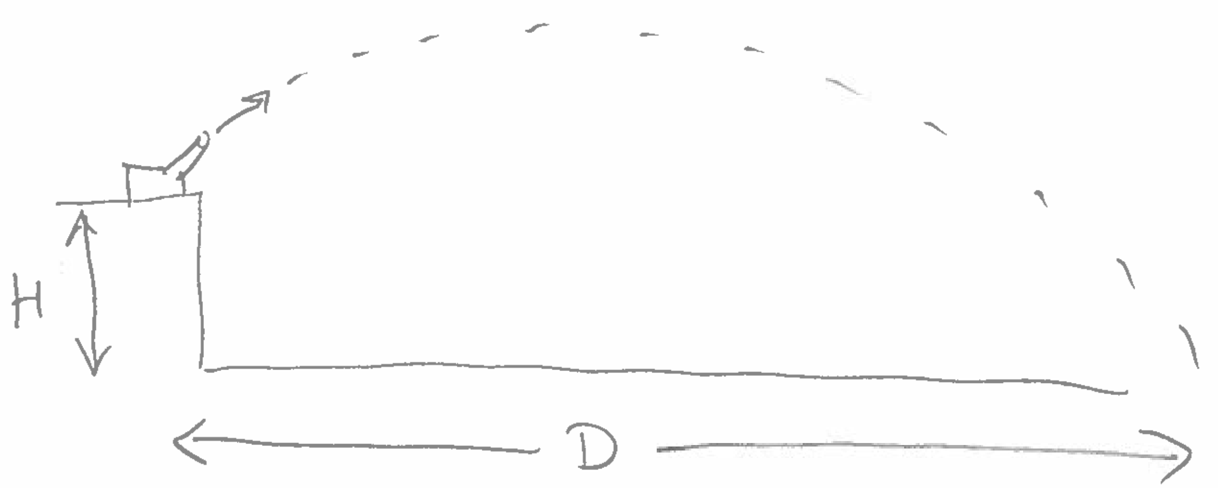
$$t_{air} = 2t_{peak} = 81.6\ s$$

Now, the shell travels in x-dir at constant speed (neglecting air resistance). So

$$L = V_x t_{air} = (V_0 \cos\theta)(2t_{peak})$$

$$= (800\ \tfrac{m}{s})\cos(30^\circ)(81.6s)$$

$$= \underline{56{,}500\ m}$$



Smith moves his gun to the top of a nearby hill, $H = 1300\ m$ above the plain. What is the new range of the gun?

This time, finding time in the air requires more work.

$$y(t) = H + V_y t - \tfrac{1}{2}gt^2$$

$$= H + V_0 \sin\theta\, t - \tfrac{1}{2}gt^2$$

The height $y(t) = 0$ when shell falls to ground.
So, find the time at which $y(t) = 0$

$$0 = H + V_0 \sin\theta \, t - \tfrac{1}{2} g t^2$$

Quadratic equation in t.

$$t = \frac{-V_0 \sin\theta \pm \sqrt{V_0^2 \sin^2\theta + 2gH}}{-g}$$

$$= \frac{-400 \frac{m}{s} \pm \sqrt{\left(400 \frac{m}{s}\right)^2 + 2\left(9.8 \frac{m}{s^2}\right)(1300 m)}}{-9.8 \frac{m}{s^2}}$$

$$= \frac{-400 \frac{m}{s} \pm 430.7 \frac{m}{s}}{-9.8 \, m/s^2} = \begin{cases} + & 84.76 s \\ - & -3.13 s \end{cases}$$

So time in air $t_{air} = 84.76 s$, and thus

$$D = V_0 \cos\theta \, t_{air} = \left(800 \frac{m}{s}\right) \cos 30^\circ (84.76 s)$$

$$= \underline{58{,}700 \, m}$$

Smith gains a bit over 2 km in range.

Bonus! Which angle will give the gun the largest range while it is on the hill?

When a gun is on flat ground, the answer is $45°$.
But on uneven ground, the answer can be a bit different.

In this case, if you choose some angle θ, you can

1) solve quadratic equation for time in air

2) compute horizontal distance via

$$x = V_0 \cos\theta \, t_{air}$$

Trial and error shows that for $H = 1300$ m, an angle slightly below 45° works slightly better

$$\theta = 45° \rightarrow x = 66{,}581 \text{ m}$$

$$\theta = 44.44° \rightarrow x = 66{,}593.$$

In real life, air resistance would be a large effect which would modify all these numbers.