Introduction to Special Relativity

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Introduction to Special Relativity, Measuring Time and Space in the Same Units, Intelligent Observers, Event and Space-time Diagrams

1.1 What is Relativity, and Why is it Special?

Suppose we are trying to describe the world as we see it. We would need to tell the location of objects in our world, the velocities of the objects, and how these change with time.

Consider two observers, let us call them Alfred (A) and Betsy (B). Alfred and Betsy could be at rest relative to each other, in which case they would see a constant distance and direction of one relative to the other—the vector displacement between the two would be constant. This is a pretty boring scenario.

Instead let us suppose that Betsy is moving relative to Alfred. According to Alfred, Betsy:

(a) could be moving in a straight line at a constant speed, with a changing separation but in the same direction

(b) could be moving in a straight line at a changing speed, i.e. accelerating, but moving in the same direction

(c) could be moving in a circle around Alfred, with a constant separation, but everchanging direction

(d) any of an infinite number of other possibilities

Alfred could record the location and velocity of Betsy in intervals of 1 second. Alfred would know that he is at rest with himself. Betsy would know that she is at rest with herself. She could then record the location and velocity of Alfred in intervals of 1 second.

We say that Alfred and Betsy are in *relative motion*, and *relativity* is a mechanism of determining Betsy's numbers knowing Alfred's numbers, or vice versa.

There is nothing new about this basic idea of relativity. The cavemen Akoba and Loana[1] would have known that if Akoba saw Loana in the East, then Loana would see Akoba in the west.

Fast forward to the time of Galileo who postulated a Relativity Hypothesis[2]:

> Any two observers moving at constant speed and direction with respect to each other will obtain the same results for all mechanical experiments.

By the latter half of the 19^{th} century Maxwell had shown that electricity and magnetism were interrelated by 4 equations, and that these predicted electromagnetic waves that travel at the speed of light. This posed problems for Galilean relativity (we will discuss these in a bit) and scientists like FitzGerald, Lorentz, and Poincaré found various patches that made things work better[3].

Einstein published his paper "*On the Electrodynamics of Moving Bodies*" in 1905, showing how the difficulties can be resolved by simply extending Galileo's Relativity Hypothesis from mechanical experiments to all experiments. Subsequent papers expanded on this new theory now called *Special Relativity.*

Initially we will discuss our friends Alfred and Betsy and have one move at a constant velocity (speed and direction) relative to the other. Special Relativity works equally well if one of our friends is accelerating relative to the other, as we will discuss much later in the course. Special relativity does not, however, discuss gravity, and assumes "flat space-time"—we'll discuss space-time shortly, and touch on curved space-time at the end of the course.

General Relativity is Einstein's theory that discusses gravity, and describes masses as "curving the very fabric of space." General Relativity requires a great deal of mathematical and physical sophistication, and will be left for other courses.

This course will discuss Special Relativity, including Reference Frames, Space-Time diagrams, invariant space-time intervals, Inertial and Free-floating reference frames, the Principal of Special Relativity, consequences of the theory on the measurement of distance intervals and time intervals, transformations between coordinates measured in different systems, velocity and frequency transformations, Time and space intervals for accelerated motion, space-like time-like and light-like intervals, and ending with the ideas of energy and momentum.

1.2 The Parable of the Surveyors

Spacetime Physics[4] begins with an excellent analogy involving surveyors on a flat earth. Paraphrased this is:

In the Institute of RIT the direction called

[1] From the movie *One Million Years B.C.*

[2] *Dialogue Concerning the Two Chief World Systems*, 1632

[3] http://en.wikipedia.org/wiki/History_of_special_relativity

[4] *Spacetime Physics, Second Edition*, Taylor and Wheeler, W.H. Freeman and Company 1992

North was sacred, and distances measured northward (and southward) were measured in the sacred unit, the mile. Distances eastward were measured in the not-so-sacred unit, the meter.

A surveyor named Anahita (Persian Goddess of day) does her work only in daytime. She uses a magnetic compass to find the direction North. She measures the coordinates of every important building in the Institute, relative to the center the Sentinel, and records the Northward (in miles) and eastward (in meters) coordinates of each. (See Figure 1-1 in Spacetime Physics.) Anahita is making more careful measurements of the objects measured by her grandfather Horus (yeah, I know he is Egyptian not Persian.) She is able to get readings with more significant figures than Horus.

Unbeknownst to Anahita, Nyx (goddess of night, Greek) has been doing her surveying at night, using Polaris to tell her where north is. She measures the same buildings as Anahita, starting from the same location, the center of the Sentinel. And she is checking the numbers of her grandfather Tezcatlipoca (Aztec).

One day Anahita is studying for a Physics exam and decides to pull an all nighter. When she stops for some coffee she meets Nyx, and the two begin to compare numbers. Alas, alack their numbers disagree, not by much, but definitely different. The differences would not have been obvious in the lower precision numbers of their grandfathers.

After several more pots of coffee arguing over the numbers, a mutual friend Koit (Estonian God of Dawn) stops by with a radical suggestion: make a unit conversion to make the northward distance end up in units of meters. To get North in meters, multiply North in miles by a constant $k = 1609.344$ meters/mile.[5]

The numbers in Table 1.1 show some of the records of Anahita and Nyx. Together with Koit they made a startling observation. In *Spacetime Physics* this is written

$$(k\,\text{North(mi)})^2 + (\text{East(m)})^2 = (\text{distance})^2 \tag{1.1}$$

The distance, and the $(\text{distance})^2$ are exactly the same number whether the daytime measurements of Anahita or the nighttime measurements of Nyx are used.

This is even easier if we use the distances measured in the same unit, meters.

$$(\text{North(m)})^2 + (\text{East(m)})^2 = (\text{distance})^2 \tag{1.2}$$

For example, for Stake A

$$\text{distance}^2 = (4010.1)^2 + (2949.9)^2 = 24783000\,\text{m}^2 = (3950.0)^2 + (3029.9)^2$$

where we have taken care to write the answer to 5 significant figures, the same as the numbers used in the computation.

The example works for the coordinates of each point, but we can also define the distance between two points by making the following definitions.

$$\begin{aligned}\Delta\text{N}_{12} &= \text{North}_2\text{ (meters)} - \text{North}_1\text{ (meters)}\\ \Delta\text{E}_{12} &= \text{East}_2\text{ (meters)} - \text{East}_1\text{ (meters)}\end{aligned}$$

[5]Alternately convert all measurements into miles by multiplying the easterly distance in meters by $1/k = 6.213712 \times 10^{-5}$

Table 1.1: Surveying Results

	Daytime Anahinda			Nighttime Nyx		
	East (m)	North (miles)	North (m)	East (m)	North (miles) (miles)	North (m) (m)
Sentinel	0	0	0	0	0	0
A	4010.1	1.8330	2949.9	3950.0	1.8827	3029.9
B	5010.0	1.8268	2939.9	4950.0	1.8890	3040.1
C	4000.0	1.2117	1950.0	3960.0	1.2614	2030.0
D	5000.0	1.2054	1939.9	4960.0	1.2676	2040.0

We'll call the two quantities the *North Interval* and the *East Interval.* Then

$$(\text{Displacement Interval})^2 = (\Delta N)^2 + (\Delta E)^2 \quad (1.3)$$

For example the north interval from A to D is

$\Delta N_{AD} = 1939.9 - 2949.9 = -1010.0$ m

and the east interval is

$\Delta E_{AD} = 5000.0 - 4010.1 = 989.9$ m using daytime numbers,

so the displacement interval is

$(\text{Displacement Interval})^2 = (-1010.0)^2 + (989.9)^2 = 2000000 \text{ m}^2$.

You can try the same calculation using nighttime numbers, with different values for ΔN_{AD} and ΔE_{AD}, but the same value for the displacement interval.

The advantage to using intervals is that we no longer are required to use one point as a reference (The Sentinel), but that each person can use a separate reference point from which to measure.

Together the three goddesses published their results and described them as the ANK Invariance Principle: the displacement interval defined in Equation (1.3) is *invariant*, that is it is the same for all observers.

1.3 Completing the analogy

Table 1.2 makes a comparison of the Parable's important results with the results for special relativity.

The last two rows have almost identical equations. Do you understand the difference?

Prior to the full acceptance of special relativity, people believed in absolute time but relative space. Let's make this statement more clear.

Imagine boarding an airplane in Rochester and deplaning in Seattle. For you on the plane, the distance from the place where you sat down in Rochester to the exit door in Seattle was perhaps 100 feet. For me staying in Rochester the distance between the two points is 2698 miles. The space interval is dependent on the observer, i.e. relative to the observer. This causes us no confu-

Table 1.2: Connecting Parable and Special Relativity

Parable	Special Relativity
North measured in miles	Time measured in Seconds
East measured in meters	Distance measured in meters
Conversion Factor $k = 1609.344$ m/mi	Conversion Factor $c = 2.99792458 \times 10^8$ m/s
$N(\text{m}) = kN(\text{mi})$	$t(\text{m}) = ct(\text{s})$
$E(\text{mi}) = E(\text{m})/k$	$x(\text{s}) = x(\text{m})/c$
North Interval $\Delta N = N_2 - N_1$	Space Interval (1D) $\Delta x = x_2 - x_1$
East Interval $\Delta E = E_2 - E_1$	Time Interval $\Delta t = t_2 - t_1$
(Displacement Interval)$^2 = (k\,\Delta N)^2 + (\Delta E)^2$	(Space-Time Interval)$^2 = (c\,\Delta t)^2 - (\Delta x)^2$
(Displacement Interval)$^2 = (\Delta N)^2 + (\Delta E)^2$	(Space-Time Interval)$^2 = (\Delta t)^2 - (\Delta x)^2$

sion.

For you on the airplane the flight may have taken 5 hours. For me on the ground, the flight would also have taken 5 hours. Normally we think of time intervals as not dependent on the observer, but absolute. This was assumed to be true until 1905.

A direct check of whether time intervals are relative or absolute requires very accurate clocks, clocks that were not available until 1949. Measurements with less precise clocks could not distinguish between absolute and relative time intervals. we might believe time to be absolute, but if we had very accurate clocks, we would find that the time interval measured by you on the plane and me on the ground would be different. This is one of the wonderfully weird consequences of relativity that we will explore in detail. And yes, that exact experiment, flying atomic clocks on airplanes, was done in 1975 and confirmed the predictions of relativity[6].

The final line in the table is interesting. For the parable the displacement interval is just the Pythagorean Theorem. In special relativity we use a space-time interval (STI) with STI^2 calculated by subtracting the square of the space interval from the square of the time interval. The negative sign will turn out to have profound effects.

1.4 Time in meters, Space in seconds

In the parable we had different units for northward and eastward positions. In special relativity we have different units for position (meters) and for time (seconds). In the parable, we found it more natural to express the northward and eastward positions in the same units. Likewise in Special Relativity we find it useful to have the same units for space and time.

You all know about light-years. A light year is a unit of distance equal to the distance

[6] Both Special and General Relativity were needed to analyze the results, SR because of the motion of the plane and GR since the airplane is at a different altitude that the ground where the gravitational field is different.

travelled by light in one year. Essentially we are measuring distance in a unit of time. To convert from meters to seconds we divide by the speed of light—and for convenience we will use the value 3.00×10^8 m/s.

Example Convert the following distances into time units. Use approprite prefixes rather than scientific notation. (a) 30 cm (about 1 foot) (b) 6378 km (radius of earth) (c) 1 a.u. = 93 million miles (distance to sun) (d) 30 a.u. (distance to Neptune)

(a) $\frac{0.30\text{ m}}{3.0\times10^8\text{ m/s}} = 1 \times 10^{-9}\text{ s} = 1\text{ ns}$

(b) $\frac{6378\text{ km}}{3.0\times10^8\text{ m/s}} = \frac{6378000\text{ m}}{3.0\times10^8\text{ m/s}} = 21.3\text{ ms}$

(c) First convert 93 million miles into meters. Here is a lazy way to do it: enter "93e6 miles in m" into Google and get the answer 1.50×10^{11} m. Then $\frac{1.50\times10^{11}\text{ m}}{3.0\times10^8\text{ m/s}} = 500\text{ s}$

(d) Multiply the answer to (c) by 30 to get 15000 s = 15 ks (or 4 hours, 10 min.)

Likewise we can convert time from units of seconds into units of meters.

Example Convert the following into units of meters, using the appropriate prefix and not scientific notation. (a) 0.025 s (this would be a 30 cm lead at the end of a 100 m dash) (b) 50 minutes (c) 1 day

(a) $(0.025\text{ s})(3 \times 10^8\text{ m/s}) = 7.5 \times 10^6\text{ m} = 7.5\text{ Mm}$

(b) 50 minutes = 3000 s, so $(3000\text{ s})(3 \times 10^8\text{ m/s}) = 9 \times 10^{11}\text{ m} = 900\text{ Gm}$

(c) 1 day = 86400 s, so $(86400\text{ s})(3 \times 10^8\text{ m/s}) = 2.59 \times 10^{13}\text{ m} = 25.9\text{ Tm}$

There are two ways to write equations in relativity. One uses conventional units such as m for space, s for time with an equation like

$$\text{Space} - \text{time Interval} = \sqrt{(c\Delta t)^2 - (\Delta x)^2} \quad (1.4)$$

and the other assumes that a single set of units is used so that

$$\text{Space} - \text{Time Interval} = \sqrt{(\Delta t)^2 - (\Delta x)^2} \quad (1.5)$$

You should be able to distinguish between the two based on the context of the problems being worked on. We will tend to use the same units for space and time, unless we have to match up to something in the lab.

Example We have not given any justification for the Special Relativity formulas yet, but here is how we could use them. Suppose a rocket carries a laser that it pulses periodically. For Ralph on the rocket, the space interval is zero, but for Grace on the ground the laser will have moved between flashes. Suppose we have the following data. Fill in the quantities marked ??

Table 1.3: Using the relativistic interval

	Δt_r Rckt Time Interval (ns)	Δt_g Gnd Time Interval (ns)	Δx_g Gnd Space Interval (ns)	Δx_g Gnd Space Interval (m)
(a)	??	15	??	3.6
(b)	15	??	20	??
(c)	40	58	??	??

(a) Find the ground space interval in ns, $\Delta x_g = 3.6$ m$/3 \times 10^8$ m/s $=$ 12 ns. Then use (Space-Time Interval)$^2 = (\Delta t_r)^2 - (\Delta x_r)^2 = (\Delta t_g)^2 - (\Delta x_g)^2$.

$(\Delta t_r)^2 - 0^2 = 15^2 - 12^2$ so that $\Delta t_r = 9$ ns

(b) $(\Delta t_r)^2 - 0^2 = (\Delta t_g)^2 - (\Delta x_g)^2$ so

$(\Delta t_g)^2 = (\Delta t_r)^2 + (\Delta x_g)^2$ giving $\Delta t_g = 25$ ns.

For Δx_g in meters, $\Delta x_g = (3 \times 10^8$ m/s$)(20 \times 10^{-9}$ s $= 6.0$ m

(c) $(\Delta t_r)^2 - 0^2 = (\Delta t_g)^2 - (\Delta x_g)^2$ so

$(\Delta x_g)^2 = (\Delta t_g)^2 - (\Delta t_r)^2$ giving $\Delta x_g = 42$ ns.

For Δx_g in meters, $\Delta x_g = (3 \times 10^8$ m/s$)(42 \times 10^{-9}$ s $= 12.6$ m

1.5 Intelligent Observers

Much of the confusion that arises in relativity comes from the basic meaning of observation, and we must spend some time on this seemingly trivial and boring subject[7]. Let's first carefully define some terms: object, location, instant, and event.

- Object: This includes an observer (Alan), another object (rocket), but also things like the crest of a wave. The object can be point-like and have a single location, or may be large and spread out.
- Location: This is a set of specific space coordinates (x, y, z) but the time is not defined.
- Instant: This has a definite time coordinate, t, but can apply to many different locations.
- Event: This refers to a specific location and time, that is a definite set of values for (x, y, z, t).

Consider the following items: define them as object, location, instant, event, or none of the above.

(a) The Declaration of Independence.

(b) The dot over the "i" in Benjamin Franklin's signature on the Declaration.

(c) Babe Ruth's final home run.

(d) The opening of the Olympic games in China.

(e) Your friend honks her horn.

(f) The sound travels from your friend to you.

(g) You hear the beep.

(h) Two successive beeps on your friend's horn.

What ambiguities do you need to worry about in using these terms? How can you resolve these ambiguities?

Next consider the values that two observers will see for the positions and instants of some events. Here use just ordinary non-relativistic physics, the stuff that operates in almost all of what you see.

[7] The following exercises are adapted from Rachel Scherr PhD thesis.

For now we will talk about the space separation and the time separation of two events.

1. A straight runway is 100 m long. A small explosion occurs at the east end of the runway (Event 1); 10 seconds later, an explosion occurs at the west end of the runway (Event 2). An airplane moves from west to east with speed 25 m/s relative to the runway.

   How far apart in space are the locations (i.e. what is the space separation) of the explosions:

   - in the frame of the runway? Explain.
   - in the frame of the airplane? Explain.

2. Two physics students, Alan and Beth, are shown in Figure 1.1. Alan and Beth have measured their exact relative distances from points X and Y.

   Sparks jump at the points marked X and Y. When each spark jumps, it emits a flash of light that expands outward in a spherically symmetric pattern. Alan, who is equidistant from points X and Y, receives the wavefront from each spark at the same instant.

   Answer each of the following questions for the observers listed.

   (a) Does Alan conclude that for his reference frame, the spark that jumped at point X jumps *before, after, or at exactly the same time* as the spark that jumped at point Y? Explain your reasoning.

   (b) Does Beth receive the wavefront from the spark that jumped at point X *before, after, or at exactly the same time* as the wavefront from the spark that jumped at point Y? Explain your reasoning.

   (c) Does Beth conclude that for her reference frame, the spark that jumped at point X jumps *before, after, or at exactly the same time* as the spark that jumped at point Y? Explain your reasoning.

   (d) Does Beth's answer in (b) agree with Alan's observation that he receives the sparks at he same time? If they do not, explain why they get different answers.

   (e) Do the answers of Alan and Beth about when the sparks occur, parts (a) and (c), agree? If they do not, explain why they get different answers.

3. A physics student named Alan and a beeper are arranged as shown in Figure 1.2. The beeper has just emitted a beep (the arcs show the sound progressing through space), and Alan wants to determine the exact time of the beep. However, he is a long way from the beeper and unable to travel to it.

   (a) Alan is equipped a large number of accurate meter sticks and clocks, and has a lot of friends (Alex, Alfred, Andy, . . .) who will assist him if necessary. Neither the meter sticks nor the clocks are affected by being moved.

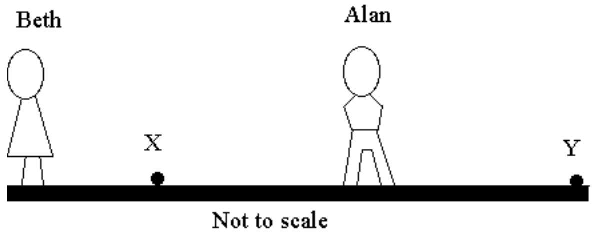


Figure 1.1: Alan, Beth, and location of sparks (X and Y). Alan is equidistant from the two sparks and receives light from the sparks at the same time.

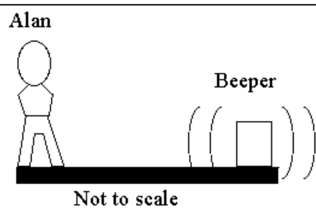


Figure 1.2: Alan and a Beeper. Alan wants to determine the time at which the beeper beeped.

i. Describe a procedure by which Alan can determine the distance to the beeper. Remember he cannot move.

ii. Describe a set of measurements by which Alan can determine the time at which the beep is emitted using his knowledge of the speed of sound in air.

iii. Describe a method by which Alan can determine the time at which the beep is emitted *without* knowing or measuring the speed of sound first. (Hint: Alan's assistants are free to stand at any location.)

(b) A fugitive from justice is at large in Rochester. His identity and exact whereabouts are unknown. A reporter has reason to believe that the fugitive will soon confess to his crime, and wishes to record as exactly as possible the time and place of the confession. Her funding for this project is excellent

i. Describe an arrangement of observers and equipment with which the reporter may record the position and time of the confession.

ii. An observer's reference frame is an arrangement of assistants and equipment with which the observer may record the position and time of anything that occurs.

Justify the claim that the reporter's arrangement of observers and equipment is the reporter's reference frame.

(c) A horn is located between Alan

and the beeper. The beeper beeps once and the horn honks once. Alan hears the two sounds at the same instant in time.

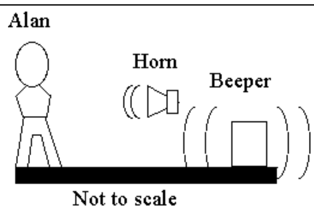


Figure 1.3: Alan hears a horn and a beeper at exactly the same time.

i. In Alan's reference frame, is the beep emitted *before, after, or at the same instant* as the honk is emitted? Explain.

ii. Describe a method by which Alan can measure the time separation between the emission of the beep and the emission of the honk in his reference frame *without* knowing or measuring the speed of sound first.

An *intelligent observer* is equipped with measuring devices (such as meter sticks, clocks, and assistants) and is able to use them to make correct and accurate observations of where and when something occurs. Intelligent observers correct for any transmission time taken by signals. All observers in the study of relativity are intelligent observers.

1.6 Events and Event Diagrams

Event diagrams are a first, very pictorial, method of showing how events occur. They are useful for 1D and 2D situations. Consider a 1D situation. Objects are spaced along a single axis that we boringly call x, and by convention x increases as we move to the right. A single drawing can show all the objects and their locations for one specific instant in time. It is a snapshot of the world.

If nothing moves, one such diagram is all we need. When objects move we need to have several snapshots at different times—i.e. we need frames of a movie.

By convention we stack the frames one above the other, so that on a piece of paper the first snapshot is at the *bottom* of the paper, and successive snapshots are placed above the first. Thus time increases as we move up the diagram. The group of frames is like a flip book.

An event can be shown by marking and labeling a particular location on a particular frame (snapshot), and the entire diagram just discussed is called an *Event Diagram.*

Example Recall the events described in the last section in connection with Figure 1.3. Draw an event diagram for this situation.

At the bottom of the page, sketch a picture showing Alan, the beeper, the horn, and any other objects of interest at the instant the beeper beeps. Indicate the location of the event "the beeper beeps" on the picture. Label it E1.

Above that picture, sketch another picture showing the objects of interest at the instant the horn honks. Indicate the position of the event "the horn honks" on this picture. Label it E2.

Above that picture, sketch picture(s) showing the objects of interest at the instant(s) of the remaining events. Sketch a separate picture for each different instant of time. Indicate the location of each event on the appropriate picture.

Be sure you understand the meaning of the diagram you just drew by answering the following questions.

Does the first frame, the picture you drew at the bottom of the page, represent an object, a location, an instant, an event, several or none of these?

Can more than one object appear in a single frame? Can more than one event occur in a single frame?

Can an object occur in several different frames? Can an event occur in several different frames?

1.7 Synchronization of Clocks

We require intelligent observers to have methods to ensure that they are measuring quantities in the same units, and with the same origin. In 1999 a Mars orbiter was lost because NASA used SI units, while Lockheed Martin used British Engineering units, and they failed to do a proper conversion. We must have a method to avoid such problems.

Setting up the coordinate system for distances is relatively easy. We lay out a grid of metersticks. We can produce them at a central plant (in order to ensure that they are identical) and disperse them throughout the universe. Figure 2-6 in Spacetime Physics shows a latticework grid.

We are confident that moving a meter stick and then bringing it back to rest will not change the length of the stick, and we will require that the sticks touch each other, so a grid is established. Suppose we are at the origin of the grid. We can then be confident that when our friend records a location on a meter stick near her, that her reading will tell us how far that location is from us.

Clocks are another matter. We can establish a calibration method for setting the tick rate of a clock so that we are confident that our friend's clock has the same tick rate as ours. But how do we synchronize the two clocks, that is how do we ensure that when our clock reads noon, her clock reads noon? Here is a method to try.

Alan and Beth are exactly 10 meters apart relative to the floor as shown in Figure 1.4. Each of them wears a watch. Both watches are extremely accurate, run at the same rate, and measure time in meters. (Remember, one meter of time is the amount of time it takes light to travel one meter.) However, the reading on Beth's watch is not the same as the reading on Alan's watch.

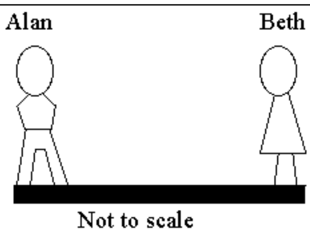


Figure 1.4: Alan and Beth Prepare to Synchronize Watches.

1. Determine the amount of time, in meters, that it will take a light signal to travel from Alan to Beth.

2. Beth and Alan decide in advance that at the instant Alan's watch reads 50 meters, Alan's laser pointer will emit a pulse of light in Beth's direction.

   What time will Alan's watch read at the instant Beth first receives the light from the laser pointer?

3. Describe a method by which Beth could synchronize her watch with Alan's (i.e. make her watch have the same reading as Alan's at every instant.)

4. Another physics student, Caroline, is at rest with respect to Alan and Beth but is very far from them. Caroline looks at the reading on Alan's watch with a powerful telescope, and finds that at every instant the reading she sees on Alan's watch through the telescope is identical to the reading on her watch.

   Is Caroline's watch synchronized with Alan's? Explain why or why not.

1.8 More on Events and Intelligent Observers

The following exercises should help you better understand events, event diagrams, and measurement by intelligent observers.

1. Two spaceships, A and B, pass very close to each other. Alan and Andy ride spaceship A, Alan at the front of ship and Andy at the rear. Beth and Becky ride ship B, with Beth at the front of her ship and Becky at the rear. According to Alan, ship B moves to the left with speed v = 3 m/s and ships A and B each have length 12 m.

   Define three events as follows:

   - Event 1: Alan and Beth are adjacent
   - Event 2: Andy and Beth are adjacent
   - Event 3: Alan and Becky are adjacent

   The first frame of an event diagram is shown in Figure 1.5.

   Define $\Delta x_{12}^{Alan} = x_2^{Alan} - x_1^{Alan}$, where the superscript tells who is making the observation, and the subscripts identify the event or events of interest. This quantity can be either positive or negative.

   Determine numerical values for the following ratios

   - $\frac{\Delta x_{12}^{Beth}}{\Delta x_{12}^{Alan}}$
   - $\frac{\Delta x_{13}^{Alan}}{\Delta x_{12}^{Beth}}$ Be sure to explain your rea-

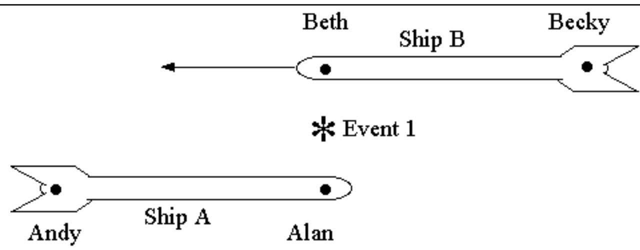


Figure 1.5: Spaceship flyby showing Event 1 Viewed by Alan

soning in words.

2. A train moves with constant nonrelativistic speed along a straight track. The train is 12 meters long.

   Alan and Andy stand 12 meters apart at rest on the track (see figure). Beth and Becky stand at rest at the front and rear of the train, respectively.

   Define these three events.

   - Event 1: Alan and Beth pass each other.
   - Event 2: Andy and Beth pass each other.
   - Event 3: Alan and Becky pass each other.

   (a) On a large sheet of paper, sketch an event diagram showing Alan, Andy, Beth, and Becky at the instants of events 1, 2, and 3 in Alan's frame. (That is, sketch a separate picture for each different instant; sketch pictures for successive instants one above the other; and indicate the location of each event on the appropriate picture.)

      i. What feature(s) of your event diagram can be used to indicate that it is a diagram for Alan's frame?

      ii. How would an event diagram for Andy's reference frame compare to the one you drew above? Explain.

      iii. What procedure could Alan

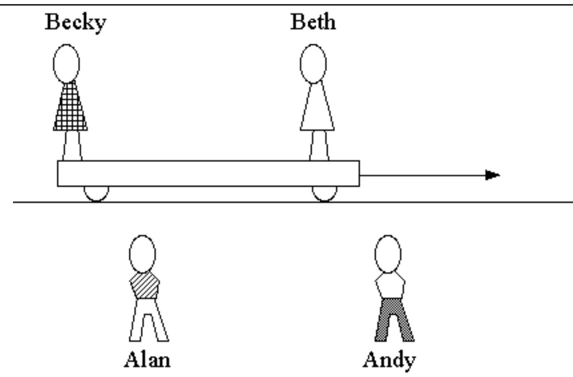


Figure 1.6: Alan, Andy, Beth and Becky making measurements for a train.

(or Andy) follow to measure the distance between the locations of events 1 and 2?

iv. How far apart in space are the locations of the following pairs of events in Alan's frame?

- Events l and 2
- Events 2 and 3
- Events 1 and 3

(b) Sketch an event diagram showing events 1, 2, and 3 in Beth's frame. Be sure your diagram correctly represents the motion of the train in this frame.

How far apart in space are the locations of the following pairs of events in Beth's reference frame?

- Events l and 2
- Events 2 and 3
- Events 1 and 3

(c) How does Beth's procedure for measuring the distance between the positions of two events compare to Alan's procedure?

(d) On the basis of your answers above, develop a general rule that uses an event diagram to determine how far apart the locations of two events are in a given reference frame.

3. Give interpretations for the magnitude of each of the following quantities; that is, tell the meaning of the number in this physical context. One has been provided as an example. Some quantities may have more than one interpretation.

(a) Δx_{12}^{Alan} This is the displacement of Beth (train) as measured by Alan (Andy).

(b) Δx_{13}^{Alan}

(c) Δx_{23}^{Alan}

(d) Δx_{12}^{Beth}

(e) Δx_{13}^{Beth}

(f) Δx_{23}^{Beth}

4. A train of unknown length moves with constant nonrelativistic speed on the same track. Alan and his assistants stand shoulder-to-shoulder on the track.

   (a) Describe a method by which Alan all by himself can determine the length of the train in his frame if he knows the speed of the train in his frame. Specify two events associated with this measurement procedure.

   - Event a:
   - Event b:

   (b) Describe a method by which Alan can determine the length of the train in his frame without knowing or measuring its speed first. Specify two events associated with this measurement procedure.

   - Event c
   - Event d:

5. Suppose event 4 occurs at the front of a long ship, and event 5 occurs at the rear of the same ship. Describe the circumstances in which the absolute value of $\Delta x_{45}^{who?}$ is equal to the length of the ship—be sure to identify who is the observer in each case.

   - in the frame S in which the ship is at rest
   - in frame F, in which the ship is moving

   Draw event diagrams to support your answers.

1.9 Velocity

For most of the course we will focus on the simplest of motion, motion in a straight line (1D motion) at a constant speed. We include the speed (magnitude) and direction (we could use left/right, East/West, but most commonly $\pm$ to describe the two directions) in the velocity of an object, v.[8]

If a single observer measures a space interval Δx and a time interval Δt, then we can write

$$\Delta x = v\Delta t \tag{1.6}$$

Sometimes it is more useful to think of this as

$$v = \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \tag{1.7}$$

where the last form is the calculus version.

It is imperitive that you think of the equations as written, with intervals, and NOT as $x = v\,t$ which assumes that the starting point is the origin at $t = 0$.

Conventionally we have units like m/s when we use Equation (1.7). What happens if we measure Δx and Δt in the same units? Equation (1.7) then says that the velocity has no units. It is easier to think of this in terms of the speed as a fraction of the speed of light. In dimensionless units $c = 1$.

[8] Velocity is a vector and in 2D and 3D we will need to write it as $\vec{v}$. A two dimensional vector velocity will have components, $v_x = v\cos\theta$ and $v_y = v\sin\theta$ where v is the speed and θ is the angle with the x-axis.

1.10 From Event Diagrams to Space-Time Diagrams

Event diagrams are messy when we want to draw many objects on a diagram and indicate events. A 1D space-time diagram contains the same information in a more compact form, as well as in a quantitative form.

Consider a graph where the horizontal axis is space, x, and the vertical axis is time, t. A horizontal line represents many locations at one instant of time. A vertical line represents one location at many instants of time. **Note that this is the reverse of what you usually draw in physics.**

We use the same units for the vertical and horizontal axes. For convenience we will use the same scale so that when 5 squares = 20 ns on the horizontal axis (space), then 5 squares = 20 ns on the vertical axis (time) also.

An event is a point somewhere on this graph—it has a specific position and time coordinate.

A line on the graph represents the object at different times. Suppose the object is not moving. Then its location is constant, and on the graph we draw a vertical line at the left and right ends. **A vertical line represents zero velocity.**

What would we draw for an object moving to the right at a velocity $v = +0.2c$? First we should recognize that the proper way to give the velocity for this graph is with dimensionless units, so $\beta = +0.2$ where we use β to represent the dimensionless velocity.

Recall Equation (1.6), $\Delta x = \beta \Delta t$, where we measure time and space in the same units and speed is dimensionless. Remember that slope on a graph is rise/run, so the slope on our space-time graph is

$$\text{slope} = \frac{\Delta t}{\Delta x} = \frac{1}{\beta} \tag{1.8}$$

Thus on a space-time diagram, **steep lines are slow speeds**. You should have purchased a pad of engineering note paper that is green with a coarse grid. Each square could represent 1 ns. Make the origin somewhere in the middle of the paper. On it draw and label a horizontal axis for space, and a vertical axis for time. Then draw lines representing:

(a) Oscar, an object at $x = 1$ ns that is not moving.

(b) Ralph, an object that is located at $x = -10$ ns at $t = 0$ ns and is moving to the right at $\beta_R = +0.40$.

(c) Lorraine, an object located at $x = +15$ ns at $t = -5$ ns and is moving to the left at $\beta_L = -0.80$.

(d) A pulse of light sent to the right by Oscar at $t = 0$ ns

The lines you have drawn are called the *world lines* for the objects. Be sure you label the world lines Oscar, Ralph, Lorraine, and Light.

On the space-time diagram you just made, mark the following events. You can use the graph to determine and record the position and time for each event. Alternately you can use algebra to find the answer. The first problem is done for you.

(a) Ralph blows a bubble at $t = -5$ ns. On the graph this event occurs at $x = -12$ ns.

We can use algebra to get the space and time coordinates more precisely.

We can expand Equation(1.6) as

$$x - x_0 = \beta_R(t - t_0) = \beta_R\, t - v\, t_0 \quad (1.9)$$

or rewrite it as

$$\Delta t = \frac{1}{\beta_R}\Delta x \quad (1.10)$$

and expand to get

$$t - t_0 = \frac{1}{\beta_R}(x - x_0) = \frac{1}{\beta_R}x - \frac{1}{\beta_R}x_0 \quad (1.11)$$

So for Ralph, $x-(-10) = 0.4\,(-5) = -2$ and solving we get $x = -12$ ns.

(b) Lorraine snaps her fingers at $t = 0$ ns.

(c) Ralph and Lorraine meet.

Verify that your coordinates for the other events are correct.

1.11 Summary

a. It is convenient to measure time and space using the same units, either units of length like meters, or units of time like seconds (sometimes called light seconds.) The conversion factor is $c = 3.00 \times 10^8$ m/s. YET TO BE ANSWERED: Why use this speed, and not another speed like the speed of sound?

b. We begin by concentrating on space intervals, Δx and time intervals Δt between events.

c. In special relativity the space-time interval, is invariant, that is the same for all observers, while both the space and time intervals are relative. YET TO BE ANSWERED: Why is this true? What does this imply?

d. We talk about objects, events, locations, and instants of time in relativity.

e. We can draw either a pictorial event diagram or a more precise space-time diagram (graph) to represent objects and events.

f. A line on a space-time diagram is called the world-line. The velocity of an object can be found as the inverse of the slope of a line on the space-time diagram. Steep lines mean slow speeds.

g. All observers are intelligent observers. They have ways to ensure that clocks are synchronized, and can use friends and a reference frame to find the location and time of any event. The transmission time of a signal from the event to the observer can be corrected for, and is not the reason for the weirdness of relativity.

h. We can get a rough idea of what is happening by drawing world lines on a space-time diagram, or by doing simple algebra using Equations (1.9) or (1.11)

Chapter 2

Reference Frames, Basic Postulates of Relativity, Time Dilation, Proper Observers, Proper Lengths, Simultaneity

2.1 Inertial or "Free-float" frames

In Chapter 1, I mentioned that Special Relativity (SR) does not deal with gravity, while General Relativity (GR) does. Chapter 2 of *Spacetime Physics* deals with how to tell if we must use GR or not.

Imagine that we are doing an experiment while on an airplane that is flying at a constant altitude above the earth. We release a pencil and it falls, evidence that gravity is acting.

If the airplane suddenly enters a steep dive, things change. The *Vomit Comet* [1] deliberately enters a parabolic path so that passengers feel the effects of "weightlessness".

The airplane and its passengers are in a state of "free fall", or as Taylor and Wheeler prefer to call it, "free float". For the duration of the parabola the passengers cannot distinguish their environment from a truly gravity-free environment. The space shuttle is likewise a free float frame of reference, continuously falling around the earth in a circular orbit.

The free float frames that we have discussed are local frames, and if we look carefully enough, we can tell whether a gravitational force acts. Consider an initially horizontal railway car dropped so that it appears to be a free float frame as shown in Figure 2.1(a).

Suppose you are inside the car and watch two balls, one at each end of the car. For some period of time you feel no effects of the earth—that is until the car hits the ground.

[1] Or *Weightless Wonder* as NASA prefers to call it, http://en.wikipedia.org/wiki/Vomit_comet About 1/3 of the passengers get violently ill, 1/3 feel mild distress, and 1/3 are fine.

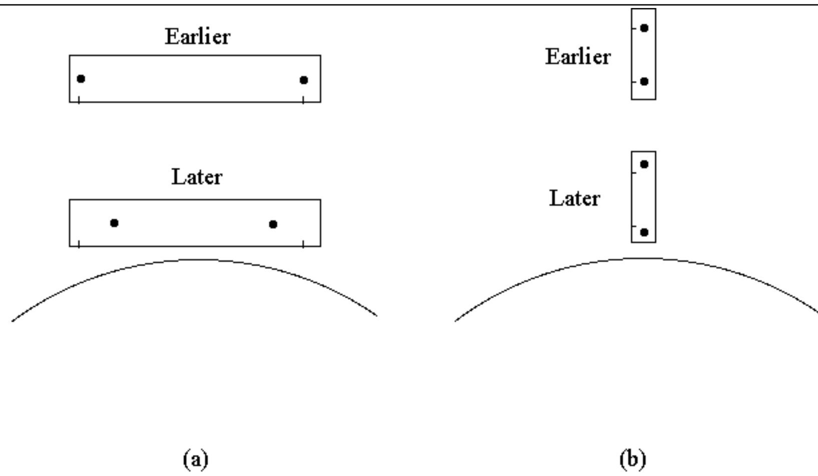


Figure 2.1: A large railway car dropped near the earth (a) horizontally (b) vertically. A ball is located at each end, and is free of the railway car. If the frame is NOT inertial (free-float, Lorentz), the separation of the balls will change during the duration of the experiment. This depends on the care with which we can take measurements.

Ouch!

Careful measurements of the positions of the balls reveals that they come closer together during the duration of the fall. Likewise in the case of the vertical fall, Figure 2.1(b), careful measurements reveal that the balls move farther apart. There appears to be some mysterious force that brings the two balls closer together when the car is horizontal, and farther apart when it is vertical.

This is the result of a non-uniform gravitational field around the earth. In Figure 2.1(a) the balls fall toward the center of the earth, while in Figure 2.1(b) the ball initially farther away from the earth moves less than the ball closer to the earth due to the variation of gravitational force with distance. (Here I have used the language of classical physics and not the language of GR.)

With this backdrop we can describe a local free float frame of reference. **The reference frame is said to be "free float" or "inertial" or "Lorentz" providing that in a certain region of space and for a certain amount of time, throughout the region, and within the measurement error, Newton's First Law is valid: i.e. a free particle at rest remains at rest, and a free particle having a velocity maintains that velocity, both size and direction.**

What does this mean in practice. If the railway car in Figure 2.1(a) is 20 m long and is dropped from a height of 315 m with no air resistance, then it will hit the surface in 8 s = 2.4×10^9 m of time. During the fall the balls will move toward each other a distance of 1 mm. So the car can be considered an inertial frame providing we focus on only 8 s of its world-line and make measurements of distances only to the nearest cm.

If it is dropped as in Figure Figure 2.1(b) with the lower edge of the box 315 above the surface, then in the 8 seconds before it hits the earth the balls will separate by 2 mm. Again the car can be considered an inertial frame providing we focus only of 8 s of its world-line and make measurements of distances only to the nearest cm.

2.2 What is the Frame of a Reference Frame

In Chapter 1 we described a method of setting up a grid of metersticks throughout the universe, and a method of synchronizing clocks for observers in a single reference frame.

Suppose that we have a local inertial reference frame "A", Albert, and inside this local frame is a second observer "B", Betsy, moving with a constant velocity with respect to A. Then B will also be a local inertial reference frame subject to the same constraints on time and accuracy of measurement as the first. (In the examples that were given in the last section this was for 8 sec and position accurate to the nearest cm.

Albert and Betsy will each have a lattice of meter sticks and clocks at rest with himself or herself. They will make measurements using their own meter sticks and clocks. Special Relativity will allow us to compare the readings that the two observers make.

2.3 Einstein's Postulates

Einstein's Postulates of Special Relativity are generally given as

1. All Laws of Physics are the same in all free float inertial reference frames. E.g. $\sum \vec{F} = m\,\vec{a}$ for all inertial frames.

2. The speed of light in vacuum is the same for all observers.

We will initially talk about two observers in different reference frames, one moving with dimensionless speed β with respect to the other ($\beta = v/c$), and we will ask about what is the same for both observers and what is different.

What is different for the two observers? Specific numbers for things like space intervals, time intervals, velocities, accelerations, forces, electric, and magnetic fields will be different. For example, Al and Becky are two observers who measure the intervals between two events. For these two events, Al measures a space interval of +51 m and a time interval of +60 m. For the same events, Becky measures a space interval of +5 m and a time interval of +32 m.

What is the same? The Laws of Physics such as Newton's Laws and the electromagnetic Maxwell Equations are the same. Fundamental constants—things like Avogadro's number, the charge of the electron, and the constants ϵ_0 and μ_0 from electricity and magnetism have exactly the same numeric value in the two inertial reference frames. The events are the same—if Al sees an event of a spark jumping a gap, so does Becky.

Let's take a brief detour into electricity and magnetism. In the 1780's, Charles Augustin de Coulomb developed the law explaining electric forces. In SI units there is a fundamental constant ϵ_0 (epsilon-sub-zero), the permittivity of free space, $\epsilon_0 = 8.854 \times 10^{-12}$ $C^2/N/m^2$. In about 1820 Jean Babtiste Biot and Félix Savart found a law to explain the magnetic force that includes a constant μ_0 (mu-zero), the permeability of free space, $\mu_0 = 4\pi \times 10^{-7} N/A^2$.

Using Einstein's First postulate we know that the values of ϵ_0 and μ_0 are the same for different inertial reference frames. In 1861, James Clerk Maxwell took the equations of electricity and magnetism and showed that by combining them he could explain that light was a combination of an electric and a magnetic field, and that it should have a speed of

$$c = \sqrt{\frac{1}{\epsilon_0 \mu_0}} \qquad (2.1)$$

Using the values for the constants given above. $c = \sqrt{\frac{1}{(8.854\times10^{-12})(4\pi\times10^{-7})}} = 2.998 \times 10^8$

with units given by $\sqrt{(\mathrm{Nm}^2/\mathrm{C}^2)(\mathrm{A}^2/\mathrm{N})} = \sqrt{\mathrm{m}^2(\mathrm{C/s})^2/\mathrm{C}^2} = \mathrm{m/s}$

The First Postulate therefore says that since ϵ_0 and μ_0 are fundamental constants, they have the same value in all reference frames, and therefore the speed of light in vacuum is the same for observer's in all reference frames. Thus Einstein's Second Postulate is a specific case of the First Postulate.

2.4 Time Dilation

Einstein figured out his theories not by doing real experiments, but rather by doing thought experiments (*Gedankenexperiment* in German) and we shall begin our development of the relativity equations with a classic gedankenexperiment.

Louise (L) sits in a lab and watches Ralph (R) move to the right in a rocket with speed β. Ralph has a laser pointer that he aims at the far side of the rocket where there is a mirror. Event 1 is Ralph briefly sending a pulse of light. Event 2 is the light pulse reaching the mirror, and Event 3 is the light pulse returning to Ralph. Event diagrams as seen by Louise and Ralph are in Figure 2.2

For the sake of subsequent calculation, we can draw the three "frames" superimposed as is done in Figure 2.3.

Call the width of the rocket, between Ralph and the mirror, $W/2$. According to Ralph the pulse has travelled a total distance W before he sees it again, and the time interval between events E1 and E3 is $\Delta t_{13}^{Ralph} = W/c = W$, where we have used the fact that in dimensionless units, $c = 1$. Recall that the superscript tells who is measuring the time interval between the events, in this case Ralph.

Now consider the view of Louise. For her, Ralph and the rocket are moving to the right

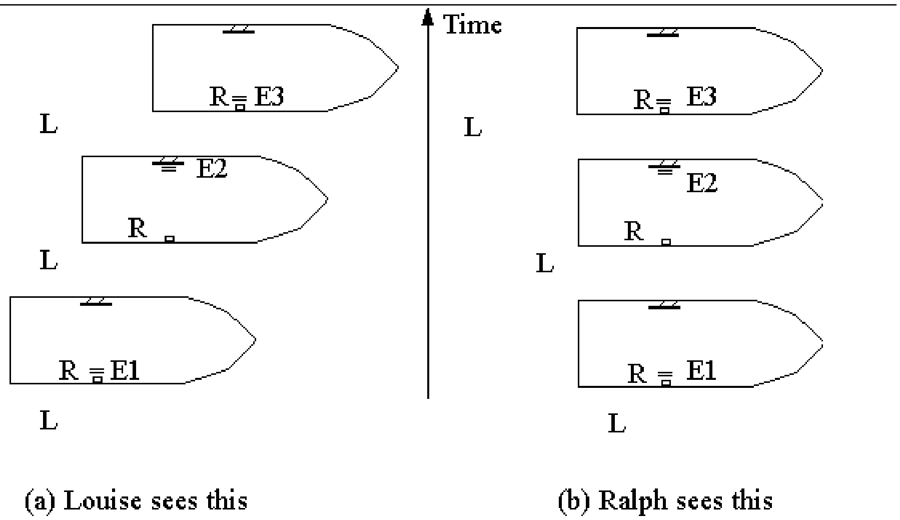


Figure 2.2: Event Diagrams for two views of a gedanken experiment, (a) Louise (L) and (b) Ralph(R). Events E1: Pulse leaves Ralph E2: Pulse reaches mirror E3: pulse returns to Ralph

with a speed β. The time interval between events 1 and 2 is half the total interval, $\Delta t_{13}^{Louise}/2$, and during this time the rocket including Ralph and the mirror have moved to the right a distance $\Delta x/2 = \beta\Delta t_{13}^{Louise}/2$. The light according to Louise has travelled along the *diagonal* taking a time

$$\frac{\Delta t_{13}^{Louise}}{2} = \frac{diagonal}{c} = diagonal. \qquad (2.2)$$

Now we use the Pythagorean equation to say

$$\begin{aligned} diagonal^2 &= \left(\frac{W}{2}\right)^2 + \left(\frac{\Delta x}{2}\right)^2 \\ &= \left(\frac{W}{2}\right)^2 + \left(\frac{\beta\Delta t_{13}^{Louise}}{2}\right)^2 \end{aligned} \qquad (2.3)$$

Combining Equations 2.2 and 2.3 and doing some algebra we end up with

$$\Delta t_{13}^{Louise} = \frac{W}{\sqrt{1-\beta^2}} = \frac{\Delta t_{13}^{Ralph}}{\sqrt{1-\beta^2}} \qquad (2.4)$$

The factor in Equation 2.4 occurs so frequently that we give it the symbol gamma. If $\beta < 1$, then $\gamma > 1$.

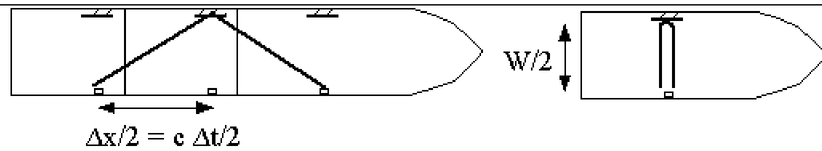
$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad (2.5)$$

and therefore we can write

$$\Delta t_{13}^{Louise} = \gamma\Delta t_{13}^{Ralph} \qquad (2.6)$$

We will use γ throughout the course. It is instructive, therefore, to calculate γ for several velocities. Fill in values in Table 2.1.

Example Suppose Ralph measures a time interval of 24 μs and according to



(a) Louise sees this (b) Ralph sees this

Figure 2.3: Multiple exposure of the Event diagram. The heavy lines are the paths taken by the pulse as seen by Louise and Ralph.

Speed β	Dilation Factor γ
0.1000	
0.3000	
0.5000	
0.6000	
0.8000	
0.9000	
0.9900	
0.9990	
0.9999	

Table 2.1: Values of γ

Louise the rocket is moving to the right at $\beta = 0.6$. What is the time interval measured by Louise?

$\gamma = 1/\sqrt{1 - .36} = 1.25$ so $\Delta t^{Louise} = 1.25(24\ \mu s) = 30\ \mu s$

Example Suppose Louise sees the rocket moving at $\beta = 0.06$. What will she measure for the time interval?

$\gamma = 1/\sqrt{1 - 0.0036} = 1.0018$ so $\Delta t^{Louise} = 1.0018(24\ \mu s) = 24.04\ \mu s$

Example Suppose Louise sees the rocket moving at $\beta = 0.99$. What will she measure for the time interval?

$\gamma = 1/\sqrt{1 - 0.9801} = 7.09$ so $\Delta t^{Louise} = 7.09(24\ \mu s) = 170\ \mu s$

Notice that reducing the speed by a factor of 10 has a huge effect on the time interval seen by Louise. For a speed of 0.06 the difference between Louise and Ralph is small, but for a speed of 0.60 it is large and unmistakable. As the speed approaches the speed of light, $\beta \to 1,\ \gamma \to \infty$, the effect is huge!

The time interval measured by Louise in this case is larger than that measured by Ralph. The common term used for the increase in the time interval is *time dilation*[2]. Louise sees Ralph's clock moving, and often time dilation is described as "Moving clocks run slow."

Special relativity contains a number of seeming *paradoxes*, that is situations that viewed from two different viewpoints appear to give contradictory results. The paradoxes allow us to sharpen our thinking, because when we think about them carefully the paradox disappears.

Consider the paradox that relates to the situation just analyzed. Louise says that Ralph moves. We have Equation 2.6 that says the time interval measured by Louise is larger than the time interval measured by Ralph. (Ralph's clock runs slow.)

[2]Or if you want to sound British, dila*ta*tion.

But what does Ralph think? Ralph sees Louise moving, and therefore would say that his time interval should be larger than Louise's time interval. (Louise's clock runs slow.)

Only one of these can be correct. How can the paradox be resolved? Can you see where the asymmetry comes from? Hint: Think about the idea of intelligent observers that we discussed earlier.

2.5 Some Nice Advanced EXCEL Features

Having to do the calculations over and over with a calculator is tiring, and prone to error. Even if you can program a special calculation pattern into you calculator, you cannot easily print the results. Excel allows more reliable calculations—at least if we program it right.

Named cells

Traditionally a cell is addressed by Column and Row such as C6. This is a relative address, meaning that if you enter a formula into cell B3 like `=C6^2`, and then copy the formula into cell C3, the formula changes to `= D6^2`. This can be very useful if we want to apply the same formula to several different values.

Sometimes however we might want to refer to the exact same cell. If cell A1 contained a value for acceleration, and we wanted to compute $c = a \times t^2$ for several different values of t in cells in column B, we could not copy the formula unless we went back and changed the reference to the acceleration. So if we typed `=A1*B2^2` into cell C2 and copied it to cell C3, it would become `=A2*B3^2`, and we would manually need to change A2 to A1, as was done in the table below.

	A	B	C
1	5.5	t	c
2		2	`=A1*B2^2`
3		4	`=A1*B3^2`
4		6	`=A1*B4^2`

Instead we could use an absolute reference, \$A\$1 so that the equation is `= $A$1*B2^2`. When this is copied, only the relative cell reference, `B2`, changes.

Even nicer would be to name the cell with a word like "accel". One way to do this is to find the cell address up in the tool bar. With the cursor in cell A1, the cell address should be A1.

Click in the cell address in the tool bar and type "accel" and enter. Now the cell has that name, and we can write an equation `=accel*B2^2`.

Writing your own functions

Excel has a wealth of functions like `sqrt, abs, sin, exp, max, average`— the whole list can be seen using `Insert Function` from the *Insert* menu or the f_x shortcut. You can add User Defined functions also.

Suppose that you want a function that will calculate γ. Here are the steps.

(a) Go to

`Tools/Macro/Visual Basic Editor` Two windows called *Projects* and *Properties* should appear.

(b) Go to `Insert/Module` and a *Workbook1-Module1 (Code)* window should open.

(c) Click in the Workbook area. The first command defines the function.

`Function gamma(beta)` which starts the definition of a function called *gamma* that will take a single argument, *beta*.

When you hit return, an End Function statement will appear.

(d) Now type the function. It can have several lines if needed, and have other variables besides beta and gamma, but somewhere you must have gamma = ...

For this function a single line suffices, `gamma = 1 / Sqr(1 - beta ^ 2)`. Click back in the spreadsheet and see if it works.

Look at the function list and you should be able to find your function in the User Defined category.

Your test on writing functions. If you are given a value of γ, what is the value of β? First work this out algebraically, then write a function for it called *beta(gamma)*. This can appear in the same module as *gamma(beta)*.

Can I Save User-Defined Functions

If you save the Excel workbook, and user-defined functions are saved with it. To access these functions you must reopen the workbook.

2.6 Proper Observers, Space-time Interval

Equation 2.6 applies to very special situations where a time interval is measured between two events. One observer (Ralph) must be present at both events. The observer present at both events is called the *proper observer* and sees $\Delta x = 0$ and measures the smallest value for Δt.

This is consistent with the space-time interval $STI = \sqrt{\Delta t^2 - \Delta x^2}$. Extending the first example in Section 2.4 to include Sydney, whom Louise sees moving to the left at $\beta = -0.6$. The events are the sending and receiving of the flash in Ralph's rocket. Ralph is the proper observer of time.

Observer	Δt	Δx	STI
Ralph $\beta = 0.6$	24	0	24
Louise	30	18	24
Sydney $\beta = -0.6$	51	45	24

Table 2.2: Observations from example of previous section. Values in μs.

The data in the table show that different observers will see different values for the time interval and the space interval, but the same invariant value for the space-time interval (STI).

According to Louise, Sydney has $\gamma^S = 1.25$. Can we use Equation 2.6 to relate the time intervals of Louise and Sydney? If you try the numbers you see that

$\Delta t^S = 51\ \mu s \neq \gamma^S \times \Delta t^L = 1.25(24) = 30\ \mu s.$

Evidently Equation 2.6 must ONLY be used when one of the observers is the proper observer.

Example Louise sees Ralph move at $\beta = 0.9$ and measures a time interval between two events of $\Delta t^L = 50$ ms. Ralph is the proper observer of the two events. What does Ralph measure for the time interval?

First method Calculate $\gamma = 2.294$. Then $50 = 2.294(\Delta t^R)$ and we solve to get $\Delta t^R = 21.8$ ms.

Second Method Louise sees Ralph move $\Delta x^L = \beta t = 0.9(50) = 45$ ms. Equating the expressions for intervals, $50^2 - 45^2 = (\Delta t^R)^2 - 0$ so $\Delta t^R = 21.8$ ms.

Both methods give the same results, as they must.

Example Suppose we look at events E1=Ralph briefly sending a pulse of light and E2 = light pulse reaching the mirror in our Louise/Ralph scenario. Who, if anyone, is a proper observer of the time interval?

2.7 But how do we know transverse width is the same for Louise and Ralph?

In the derivation of the time dilation formula, Equation 2.6 we assumed that both Louise and Ralph measured the same width for the rocket, the dimension transverse (perpendicular) to the relative velocity. How can we make this assumption?

The proof of this relies on the technique of *reductio ad absurdum.* We will make an assumption, follow it to its logical conclusion, and if that conclusion is absurd we will reject the original assumption.

Assume that the lateral dimension of a moving object is less than its lateral dimension when at rest.

Imagine that we stand next to railway tracks watching a train riding on tracks heading straight away from us (into the page.) When the train is at rest, its wheel base just matches the track spacing, Figure 2.4(a). Now suppose the train is moving into the paper, and therefore shrinks in size as shown in Figure 2.4(b). This is weird, but we expect weird from relativity.

Finally imagine that we are on the train. We would see the train at rest and the tracks moving out of the paper. By the original assumption we would then see the spacing of the track reduced as shown in Figure 2.4(c).

In Figure 2.4(b) the car would leave marks inside the tracks, in Figure 2.4(c) the car would leave marks outside the tracks. We cannot have a single set of marks both inside and outside the tracks—this is absurd. So we must reject the original assumption. We could try a new assumption that the later dimension of a moving object increases, but it would also lead to an absurd conclusion.

We are left to conclude that there is no

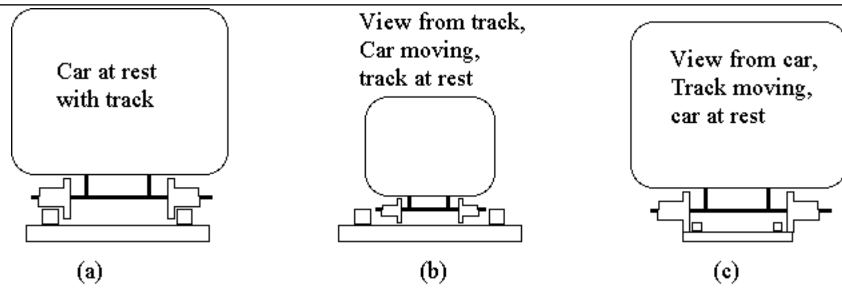


Figure 2.4: Railroad car on tracks (a) Both at rest (b) Tracks at rest, car moving into paper (c) Car at rest, tracks moving out of paper. Diagrams are drawn as if the transverse dimension changes with motion. The text shows that this assumption is absurd, and that the transverse dimension must remain unchanged as measured by different observers.

change in lateral dimension of a moving object.

2.8 Lengths of Objects Along Direction of Motion

Consider a rigid object. It has a definite length that does not change with time. How do we go about measuring its length?

If the object is at rest with respect to us we can walk to the left end and make note of its position on our reference frame of meter sticks. Then we can walk to the right end and note its position, then we subtract the two readings to get a length. It does not matter that the measurements are made at different times. Alternatively we could get a friend and ask them to stand at the right end while we stand at the left end, and each record the positions, at the same time or different times.

If the rigid object is at rest with respect to us we say that we measure its *proper length.*

What will be the length of the object as measured by intelligent observers in Ralph's reference frame?

Suppose that Louise sees a rigid object of proper length L^{Louise} and sees Ralph move to the right at speed β. The events of interest are E1: Ralph at left end of object and E2: Ralph at right end of object.

What is the time interval between the events? Louise uses numbers obtained in her reference frame and calculates the time of the trip to be

$$\Delta t^{Louise} = \frac{L^{Louise}}{\beta} \tag{2.7}$$

Ralph says that Louise is moving to the left at $-\beta$, and that he measures the proper interval between the two events (Ralph is present at both) of Δt^{Ralph}. In this time Louise has moved a magnitude of

$$L^{Ralph} = |-\beta \Delta t^{Ralph}| \tag{2.8}$$

But we know from the time dilation formula that $\Delta t^{Louise} = \gamma \Delta t^{Ralph}$. Combining Equations 2.7 and 2.8 we get the length contraction formula,

$$L^{Ralph} = \frac{1}{\gamma} L^{Louise} \tag{2.9}$$

This formula says that moving rods are measured to be shorter than their rest length.

Example An electron enters a tube traveling at $\beta = 0.90$. In the lab the tube is 100 m long.

(a) What is the length of the tube as measured by an observer moving with the electron?

The tube is at rest in the lab, so the lab measures proper length. We first calculate $\gamma = 1/\sqrt{1-0.9^2} = 2.294$.

Then $L^{Improper} = 100$ m$/2.294 =$ 43.6 m.

(b) For the lab, how long does it take for the electron to pass through the tube?

$\Delta t^{Lab} = L^{Lab}/\beta = (100 \text{ m})/0.9 =$ 111 m

(c) For the electron, how long does it take for the tube to pass completely over it?

$\Delta t^{electron} = L^{electron}/\beta =$ $(43.6 \text{ m})/0.9 = 48.4$ m

I can check using the relation between proper and improper time:

$\Delta t^{Lab} = \gamma\, \Delta t^{electron} =$ $(2.294)(48.4) = 111$ m It agrees with (b)

Example Louise in the lab has a tube at rest with respect to her that she measures to be 60.0 m long. She sees Albert moving. Albert measures the time for the tube to pass over him to be 0.267 μs. What is the relative speed of Albert and Louise?

Albert measures $\Delta t^{Albert} = 0.267 \mu$s. Is this proper or improper time?

Louise measures $L^{Louise} = 60$ m. Is this proper or improper length?

A naïve approach would be to just divide these two numbers,

$v^{Wrong} = 60.0\ m/0.267\ \mu$ s $= 2.25 \times 10^8$ m/s, i.e. $\beta^{Wrong} = 0.75$.

This is wrong because we are using numbers from different observers. We must use numbers from a single observer.

Since β is dimensionless, we convert the time into meters, $\Delta t^{Albert} = (3 \times 10^8 \text{ m/s})(0.267 \times 10^{-6} \text{ s}) = 80.0$ m.

We can either compute β as $\beta = L^{Albert}/\Delta t^{Albert}$ or $\beta = L^{Louise}/\Delta t^{Louise}$. I'll do it with Louise. To get the time interval measured by Louise and her assistants, we use $\Delta t^{Louise} = \gamma\, \Delta t^{Albert}$.

Then

$$\begin{aligned}\beta &= \frac{L^{Louise}}{\Delta t^{Louise}} \\ &= \frac{60\ m}{\gamma\, 80.0\ m} \\ &= \frac{\sqrt{(1-\beta^2)}(60.0\, m)}{80.0\ m)} \end{aligned} \quad (2.10)$$

Square both sides and rearrange to get

$$\beta^2 = \frac{(60/80)^2}{1+(60/80)^2} = 0.360 \quad (2.11)$$

so that $\beta = 0.600$

How do we actually measure the length of a moving object? It is important to understand this clearly. Consider the object at rest as shown in Figure 2.5(a). Events E1 and E3 are for recording the position of the left end of the rod, while events E2 and E4 are for recording the position of the right end of the rod. While the times will be different for E1 and E3, the position will be the same. We can use either of the even events combined with either of the odd events to determine the length. In symbols

$$L^{rest} = (x_4 - x_1) = (x_2 - x_3) = (x_4 - x_3) = (x_2 - x_1) \quad (2.12)$$

Now consider a moving rod, Figure 2.5(b). From the diagram it should be clear that the length of the moving rod is

$$L^{moving} = (x_2 - x_3) = (x_4 - x_5) \quad (2.13)$$

To measure the length of a moving object we must measure the two ends at the same time, i.e. ***simultaneously.***

Space intervals like $(x_3 - x_1)$ and $(x_4 - x_1)$ are certainly valid intervals, and may be of interest in specific applications, but they do NOT represent the length of the rod.

Also be sure you know the difference between what we measure, using our friends throughout the universe, and what we see based on light entering our eye. We will discuss this again in terms of space-time diagrams.

2.9 Simultaneity

Now for the most confusing part of relativity, the relativity of simultaneity. What you will deduce is that two events that occur at the same time but at different locations according to one observer will NOT be simultaneous for a second observer who moves relative to the first.

Example Al and his friends Aaron and Andy are standing equally spaced along the ground, with Aaron to the left, then Al, then Andy on the right, as shown in Figure 2.6. They are at rest with each other, and 50 meters apart. Immediately in front of Al is a bulb that is flashed at t = 0. This is event E1. Some time later Aaron sees the flash, event E2, and Andy sees the flash, event E3.

(a) On the top half of engineering paper draw a space-time diagram as seen by Al. Show labeled worldlines for each of the three men, and for the light that moves left and right from the bulb. Use a dashed line for the worldline of light. Label the events E1, E2, and E3.

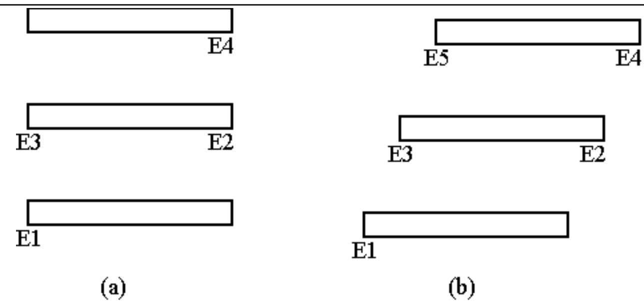


Figure 2.5: Event diagrams for measuring the length of a rod (a) at rest (b) moving. Various events are marked.

(b) According to Al, who sees the light from the bulb first, Aaron Andy, or they see it at the same time? Explain with reference to your spacetime diagram..

(c) At the time that Al flashed the bulb, Beth was at his location and moving to the right at $\beta = 0.6$. Draw and label the worldline for Beth on the spacetime diagram.

(d) Now draw an event diagram as seen by Beth on the bottom half of the paper. Here are the steps.

Calculate γ in this case, and then find the separation of Aaron, Al, and Andy as seen by Beth. Use the length contraction formula.

(e) Draw and label the worldlines for Beth, Al, Aaron, and Andy on the diagram.

(f) Finally, add worldlines for the light that moves left and right. Label events E1, E2, and E3 on the diagram.

(g) According to Beth, who sees the light from the bulb first, Aaron Andy, or they see it at the same time? Explain.

What is up with this? Can it really be true? Could we actually imagine measuring this?

Example Alan, Andy, Alex, and the usual bunch of guys are riding on Spaceship Alpha. Beth, Beatrix, Betsy, and the rest of the ladies ride on Spaceship Beta. (Will they ever meet?) Alan and Beth stand at the center of their respective ships. The spaceships may or may not be the same length—you will find out shortly. The ships are transparent so light can pass through to the observers.

Event E1: When the front end of Alan's ship is at the same location as the rear of Beth's ship, a spark jumps between them and light waves spread out. Char marks from the spark are left on both spaceships.

Event E2: Likewise when the rear of

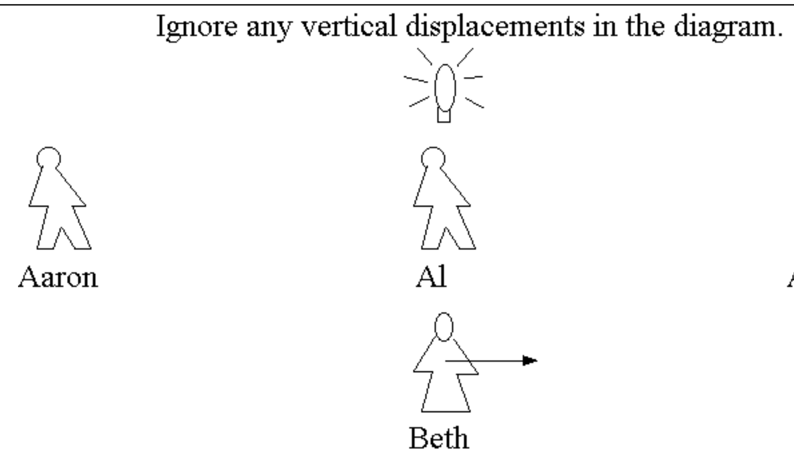


Figure 2.6: Basic Set-up for simultaneity example

Alan's ship and the front of Beth's ship pass, a spark jumps between them emitting a light pulse, and leaving char marks.

Event E3: Alan is equidistant from the char marks left on his spaceship, and receives the pulses of light from both sparks at the same time (simultaneously). Figure 2.7 shows this for the instant of time when the wavefronts arrive at Alan.]

A) 1. In Alan's frame, does spark 1 (from event E1) jump before, after, or at the same time as spark 2 (from event E2)? Explain your reasoning.

2. Alan's friend Andy stands near Char mark 2 (in their ship). According to Andy, does spark 1 jump before, after, or at the same time as spark 2 (on the right of picture)? Explain your reasoning.

3. What does the distance between the char marks in Alan's ship tell you about Beth's ship? Explain your reasoning.

4. The diagram below represents the situation in Alan's frame a short time after the sparks. Show the wavefronts at this time.

B) 1. On gridded paper, draw the space-time diagram as seen by Alan. Time and distance will be measured in the same units, "squares." Assume that Alan measures the length of Spaceship Alpha to be 20 squares. Alan sees Beth move to the right at $\beta = 0.60$. On your diagram identify events E1, E2,

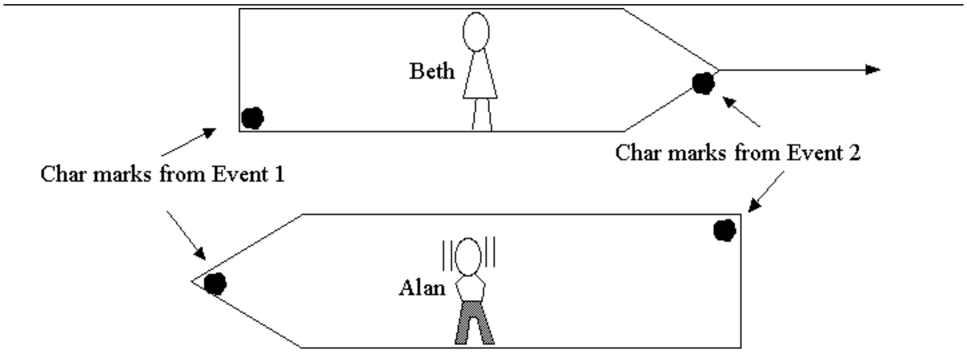


Figure 2.7: Wavefronts have just arrived at Alan

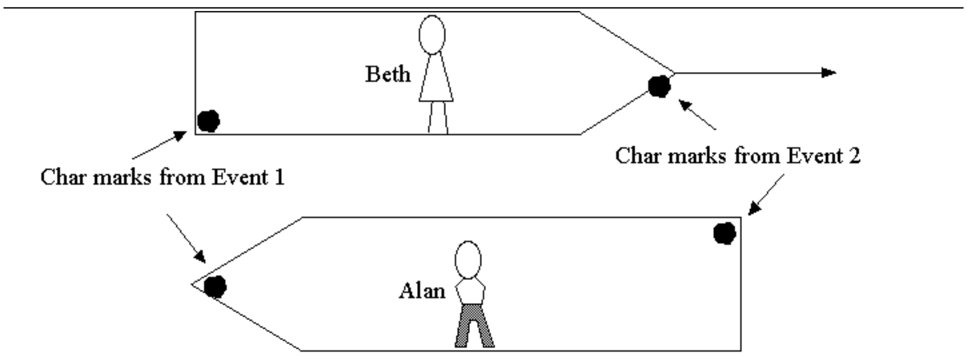


Figure 2.8: Sparks have just occurred—not to scale!

E3, and E4, Alan and Beth are exactly opposite each other.

2. Draw and label the world lines for the char marks and for the wavefronts.

3. What is the time separation, as measured by Alan, in squares, between

   Event 1 and Event 2

   Event 1 and Event 4

   Event 1 and Event 3

   You can get the answer either from algebra or from the diagram.

C) 1. Who measures the proper length of Spaceship Alpha—Alan, Beth, both, or neither. Explain

2. Who measures the proper length of Spaceship Beta—Alan, Beth, both, or neither. Explain

3. Alan measures the length of

Spaceship Alpha to be 20 squares. What does Alan measure for the length of Spaceship Beta? Explain.

4. What does Beth measure for the lengths of Spaceship Alpha and Spaceship Beta? Explain.

D) Now use another piece of paper and draw the space-time diagram as seen by Beth.

Start about 10 squares up from the bottom of the paper, and draw the diagram corresponding to Event 4. Show the locations of the front, back, and center of both ships, and draw world lines for the front, back, and center of both ships. Also show the world lines of the light emitted from the sparks.

Extend the lines as necessary and determine the times and locations of events E1, E2, and E3. Mark the events on the diagram.

Using the diagram, and using algebra, determine the time separation, in squares, between

Event 1 and Event 2, Event 1 and Event 4, Event 1 and Event 3

E) Carefully draw an event diagram as seen by Beth—that is draw pictoral diagrams like those on the first page. Time is positive upwards. Show all 4 events, as well as the spaceships and the wavefronts.

One Final Example Does the Lack of Simultaneity make sense? It probably won't make sense in terms of being a natural way of thinking, at least not now, but is it at least consistent? We have the same situation as before with Alan (and friend's) and Beth.

A. A CD burner sits at Beth's feet. In Alan's frame, when the wavefront from spark E2 reaches the burner, it starts to burn a legal copy of an Eric Clapton CD. When the wavefront from E1 reaches the burner it shuts off. If the wavefronts reach the burner at the same time, or the wavefront from E1 reaches the burner first, no music is recorded.

   In Alan's frame, is there any music burned onto the CD?

   In Beth's frame, is there any music burned onto the CD? Explain how your spacetime diagrams are consistent with this.

B. Is your answer consistent with your answers to the following question?

   Later in the day the CD is removed from the burner, and Beth travels with it via courier rocket to Alan. They play the CD. Will they hear Clapton or not?

C. Alan also has a CD burner that operates just like Beth's. In Alan's frame, is there any music burned onto his CD?

   In Beth's frame, is there any music burned onto Alan's CD? Explain how your spacetime diagrams are consistent with this.

D. Is your answer consistent with your answers to the following question?

Later in the day when Beth has traveled via courier rocket to Alan, they play Alan's CD. Will they hear Clapton or not?

2.10 Summary

Our observers are in two inertial frames with dimensionless relative speed β. and dilation factor $\gamma = 1/\sqrt{1-\beta^2}$

Proper time intervals are measured by observers present at both events. $\Delta t^{Improper} = \gamma \Delta t^{Proper}$

Proper lengths of a rigid object are measured by an observer for whom the object is at rest. $L^{Improper} = L^{Proper}/\gamma$

Two events that are simultaneous to one observer are not simultaneous for a different observer.

Chapter 3

Lorentz Transformations, Velocity Transformations, Doppler Shifts, Approximations

3.1 Mapping with Intervals Only

What information do we need to have in order to make a space-time diagram? Consider 1D space, i.e. all events occur along a single line. Well, we need to pick a starting point—this can be anywhere we choose, suppose it is Event A with coordinates for Louise of $x^{Louise} = 0 = t^{Louise}$.

Now we have another event, Event B. If we know Δt^{Louise} and Δx^{Louise}, then we can indicate where event B is on Louise's space-time map.

If we do not have Δt^{Louise} but rather Δt^{Ralph}, then the calculation is more difficult. If one of the observers is a proper observer of the time interval we can use time dilation to find the improper time interval, then use the invariant space-time interval to find the space interval. We would need to know the relative speeds of the two observers, β.

There are two other alternatives. We could set up coordinate systems for the different observers and give a way to transform positions and times of events (NOT intervals) measured by one observer to positions and times measured by a second observer. We will discuss *Lorentz Transformations* in the next section.

Another approach is to record the invariant space-time intervals between events—not position and time separately— and use those invariant quantities to map the events. We will defer detailed discussion of this, but will mention the equivalent situation for the case of the Parable Surveyors of Chapter 1.

Suppose we know the distance between all points in our world, but NOT the directions. Can we make an accurate map?

The invariant quantity for our two surveyors Anahinda and Nyx was the distance between points. Suppose we have the following data.

To → From↓	Rochester	Philadelphia	Boston	Montreal
Rochester	0	254	335	329
Philadelphia		0	273	461
Boston			0	283
Montreal				0

Table 3.1: Distances in miles between cities. Notice that no directions are given.

If the distance between two points is d, and we use one point as the origin, then the other point lies somewhere on a circle of radius $d = \sqrt{x^2 + y^2}$. Circles are easy to draw.

Let's arbitrarily pick Rochester as the origin. Philadelphia is 254 miles away. so we draw an arc of a circle of radius 2.54 inches (Scale factor 1 inch = 100 miles) and mark Philadelphia somewhere on that arc (it doesn't really matter where on the arc.)

Now to locate Montreal. Draw an arc of radius 3.29 inches (329 miles) with Rochester as the center, and an arc of radius 4.61 inches (461 miles) with Philadelphia as the center. The two arcs intersect in two locations, and we arbitrarily pick one of them and mark the location of Montreal.

Finally to locate Boston, draw arcs with Rochester (radius = 3.29 inches), Philadelphia (radius= 2.73 inches), and Montreal (radius = 2.83 inches) The three circles meet at a single point, Boston.

The procedure is shown in Figure 3.1.

What would have happened if we had used the other intersection of arcs to locate Montreal? The resulting map would be topologically equivalent—if we located Montreal above Rochester, and looked at the map in the mirror it would look the same as Figure 3.1.

In the case of Special Relativity the invariant is the space-time interval, $STI = \sqrt{\Delta t^2 - \Delta x^2}$, and this is the equation of an hyperbola. If we had a table of several events and the STI between all pairs of events we could draw a space-time diagram by using hyperbolas rather than circles . These are much harder to draw, but conceptually the idea is the same as for the Parable of the Surveyors.

3.2 The Lorentz Transformation

Often it is much simpler to define our reference system, including directions, right at the start, and then record positions and times of the events. From the positions and times it is easy to get the space and time intervals as well as the invariant space-time interval between any two events.

We consider two observers Ursula the Unprimed and Pete the Primed who are both are in inertial reference frames. All measurements made by Ursula will be unprimed (x, y, z, t, β) and all measurements made by Pete will be primed (x', y', z', t', β').

Since Pete and Ursula are both in inertial reference frames, they must have a constant relative velocity—size and direction—and we choose to call this line of relative motion the x-axis, with the x'-axis parallel

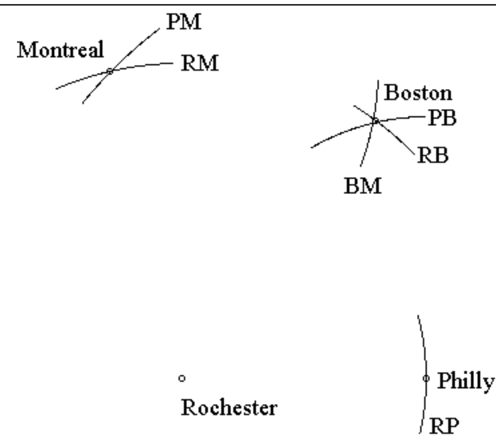


Figure 3.1: Making a map knowing only the distances between cities. The arcs are labeled with the two cities involved, e.g. RP = Rochester-Philadelphia distance.

to this. Perpendicular to x are y and z axes (and parallel y' and z' axes).

According to Ursula, Pete is moving with a velocity β along the x-axis. If β is positive, Pete is moving to the right, if β is negative Pete is moving to the left.

If Ursula sees Pete move to the right, Pete sees Ursula move to the left. We can say that Pete sees Ursula move with velocity along the x-axis of

$$\beta' = -\beta. \tag{3.1}$$

Ursula has an origin somewhere, as does Pete. We ask Pete to choose the origin of his coordinate system to lie along Ursula's x-axis. (It is moving of course, according to Ursula. According to Pete it is Ursula and her coordinate system that are moving.) Let us define the event E0 = Origins coincide.

Let's also ask Pete and Ursula to set their respective clocks to a time of zero at this event. Since we are talking about a single event, we can do all of these things without worrying about relativity. Event diagrams as seen by Ursula and Pete are shown in Figure 3.2.

The discussion so far means that an event occurring at the origin at time zero has the same coordinates in both reference frames. That is

$$x_0 = y_0 = z_0 = t_0 = 0 = x'_0 = y'_0 = z'_0 = t'_0 \tag{3.2}$$

Suppose Ursula and Pete look at a ball that can be moving and accelerating. A single event has coordinates, (x, y, z, t) and (x', y', z', t') as measured by Ursula and Pete. We want a method to take Ursula's results and figure out Pete's results, a way to transform from one observer to the other.

Consider the directions y and z that are transverse to the motion. In Chapter 2 we

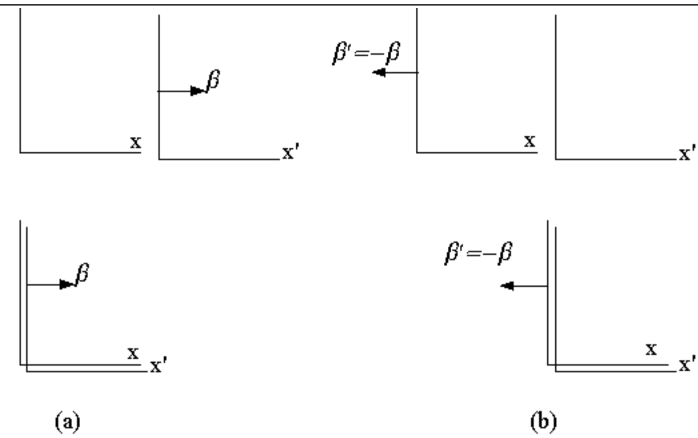


Figure 3.2: Coordinate systems of Ursula the Unprimed and Pete the Primed as seen by (a)Ursula and (b) Pete. They are drawn as event diagrams with time increasing as you move upwards. The lower diagrams are for $t = t' = 0$, and the systems are shown with a slight offset so that you can see both.

argued that relative motion will not change lengths in the transverse direction, and this means

$$y' = y \text{ and } z' = z \tag{3.3}$$

To get the transformations for x and t will take more work. First lets look at the **non-relativistic** transformation between two observers—i.e. Galilean relativity. The two observers will have coordinate systems that coincide at $t = 0$ and they both look at a ball. At time t this is shown in Figure 3.3.

From Figure 3.3 we can easily see that the NON-RELATIVISTIC transformation is

$$\begin{aligned} t'_1 &= t_1 \\ x'_1 &= x_1 - \beta t_1 \\ y'_1 &= y_1 \quad \text{Non} - \text{relativistic} \end{aligned} \tag{3.4}$$

All of these equations are linear relations, that is $x, y,$ and t occur raised to the first power only, and we never see cross terms containing the products like $x\ t$. We will require this of our relativistic transforms also.

$$t = B\, x' + D\, t' \tag{3.5}$$

$$x = G\, x' + H\, t' \tag{3.6}$$

Situation 1: Consider these two events, E1: Pete sends a spark at $x' = 0, t' = 0$. Ursula will record $x = 0, t = 0$ for this event. E2: At a later time Pete sends a second spark, $x' = 0, t' = t'$.

Pete sees the proper time interval between these events, $\Delta t' = t' - 0$. Ursula sees an improper time interval $\Delta t = t - 0 = \gamma\ \Delta t'$. Hence for this situation $t = \gamma\ t'$ and we can say that $D = \gamma$ in Equation 3.5.

Also we can write the location of E2 for Ur-

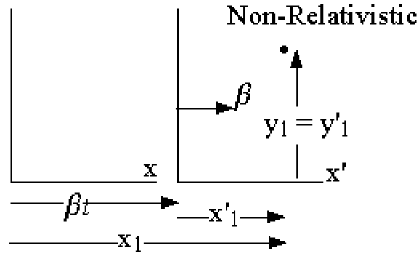


Figure 3.3: Galilean ("non-relativistic") Transformation

sula, $x = \beta\ t$ and substituting $t = \gamma\ t'$, we have $x = \beta\ \gamma\ t'$. This expression is a mixed expression—on the right we have time measured by Pete, but relative velocity measured by Ursula. Use Equation 3.1 to make all terms on the right quantities measured by Pete to get for this situation $x = -\beta'\ \gamma\ t'$, meaning that in Equation 3.6, $H = -\beta'\ \gamma$.

Situation 2: To get the other two constants we consider a third event E3 that has coordinates (x, t) for Ursula and (x', t') for Pete. Consider the interval for E1 to E3. Then $\Delta x = x - 0$, and $\Delta t = t - 0$. We use our established fact that the space-time interval is invariant to write

$$\Delta t^2 - \Delta x^2 = \Delta t'^2 - \Delta x'^2 \tag{3.7}$$

and using Equations 3.5 and 3.6,

$$(B\ x' + \gamma\ t')^2 - (G\ x' - \beta'\ \gamma\ t')^2 = t'^2 - x'^2 \tag{3.8}$$

Expand the squares and collect like terms to get

$$\begin{aligned}\left(\gamma^2(1-\beta'^2)\right) t'^2 + 2\ \gamma(B + G\ \beta')x't' \\ + (B^2 - G^2)x'^2 \\ = t'^2 - x'^2\end{aligned} \tag{3.9}$$

In the first term we have $\left(\gamma^2(1-\beta'^2)\right) = 1$ by the definition of γ, and we can cancel the t'^2 from both sides of the equation. Hence Equation 3.9 becomes

$$2\ \gamma(B + G\ \beta')x't' + (B^2 - G^2)x'^2 = -x'^2 \tag{3.10}$$

This is not an ordinary equation that is true for one particular choice of (x, t) but must be true for all values[1] of (x, t) chosen independently of each other. This means that

$$\begin{aligned}0 &= 2\ \gamma(B + G\ \beta') \\ B &= -G\ \beta'\end{aligned} \tag{3.11}$$

and that

$$(B^2 - G^2) = -1 \tag{3.12}$$

Combining Equations 3.11 and 3.12 we end up with $G = \gamma, B = -\gamma\ \beta'$ making our final transformation equations

$$t = \gamma(t' - \beta'\ x') \tag{3.13}$$

and

$$x = \gamma(x' - \beta'\ t') \tag{3.14}$$

Why are my Lorentz transformations different than the ones in the text?

[1] "For all" is denoted by ∀ in mathematics.

The text chooses to only use β, Ursula's measure of the relative velocity, while I use both β and β'. The text must then define two transformations, the "Lorentz Transformation" and the "Inverse Lorentz Transformation". By using the two relative velocities β and β', I need only one transformation.

Table 3.2: Text and Lindberg Transformations Compared

	Text		Lindberg
L10a	$t = \gamma\, \beta\, x' + \gamma\, t'$	3.13	$t = \gamma\, t' - \gamma\, \beta'\, x'$
	$x = \gamma\, x' + \gamma\, \beta\, t'$	3.14	$x = \gamma\, x' - \gamma\, \beta'\, t'$
L11a	$t' = -\gamma\, \beta\, x + \gamma\, t$	3.13	$t' = \gamma\, t - \gamma\, \beta\, x$
	$x' = \gamma\, x - \gamma\, \beta\, t$	3.14	$x' = \gamma\, x - \gamma\, \beta\, t$

In my version of the transformations, there is no change of sign between the forward and inverse transformations, all I need to do is keep all unprimed variables together, and all primed variables together. Also space and time appear symmetrically so that it is easy to get the second equation from the first just by switching the roles of x and t.

At some time you may want to define an Excel function `lorentz(beta,a,b)` that does the transformation[2]. Only one function is need in my formulation, since $t' = lorentz(beta, t, x)$, $x' = lorentz(beta, x, t)$, $t = lorentz(beta', t', x')$, $x = lorentz(beta', x', t')$,

[2] The same form of transformation transforms energy and momentum from one frame to another.

3.3 Using the Lorentz Transformation

Example Ursula measures the coordinates of E1 to be $x_1 = 100$, $t_1 = 500$ m. Pete measures the coordinates of E2 to be $x_2' = 600$, $t_2' = 900$ m. Ursula sees Pete move to the left at 0.60.

(a) Find the coordinates of E1 as seen by Pete.

Using the conventional choice of right as being positive, we have $\beta = -0.60$ and $\gamma = 1.25$.

Then $x_1' = (1.25)(100) - (1.25)(-.60)(500) = 500$ m and

$t_1' = (1.25)(500) - (1.25)(-.60)(100) = 700$ m

(b) Find the coordinates of E2 as measured by Ursula.

$\beta' = -\beta = +0.60$

So $x_2 = (1.25)(600) - (1.25)(0.60)(900) = 75$ m

and $t_2 = (1.25)(900) - (1.25)(0.60)(600) = 675$ m

(c) Find the space interval, the time interval, and the space-time interval between E1 and E2 as measured by Ursula.

$\Delta x = (75) - (100) = -25$ m,
$\Delta t = (675) - (500) = 175$ m, and

$STI = \sqrt{175^2 - (-25)^2} = 173.2$ m

(d) Find the space interval, the time interval, and the space-time interval between E1 and E2 as measured by Pete.

$\Delta x' = (600) - (500) = 100$ m, $\Delta t = (900) - (700) = 200$ m, and

$STI = \sqrt{200^2 - 100^2} = 173.2$ m

Notice that the space and time intervals for the two observers are different, but the STI is the same.

(e) Are either Pete or Ursula the proper observer for the time interval between the events? If neither of these are observers of the proper time interval, who is?

To observe the proper time interval, $\Delta x = 0$, and this is not the case for either Ursula or Pete. Consider Daisy as the proper observer. For Daisy, $\Delta x'' = 0$. Call the velocity of Daisy relative to Ursula, β'' and use the Lorentz transformations for Events E1 and E2.

$x''_1 = \gamma_D(x_1 - \beta_D\ t_1) = \gamma_D(100 - \beta_D\ 500)$

$x''_2 = \gamma_D(x_2 - \beta_D\ t_2) = \gamma_D(75 - \beta_D\ 675)$

Equate the two, since the events occur at the same location for the observer of proper time.The γ_D's cancel, leaving an equation that is easy to solve for β_D,

$(100 - \beta_D\ 500) = (75 - \beta_D\ 675)$

$\beta_D = -25/175 = -0.143.$

Daisy must move to the left at a speed of 0.143 in order that E1 and E2 occur at the same location for her and she will be the proper observer for the time interval between these two events. The proper time interval is ($\Delta t'' = \sqrt{\Delta t^2 - \Delta x^2} = 173.2$ m.

(f) What is the velocity of Daisy as measured by Pete?

We can use the same approach as the previous part, but using Pete's data.

$(500 - \beta'_D\ 700) = (600 - \beta'_D\ 900)$

$\beta'_D = 100/200 = 0.500.$

Example Imaginary Space-Time Interval?

Ursula sees event E3 with $x_3 = 500, t_3 = 700$ m and E4 with $x_4 = 688, t_4 = 813$ m. She sees Pete move to the right at 0.60.

(a) What does Pete see for coordinates of E3?

$x_3 = 1.25(500 - 0.60(700)) = 100$ m, $t_3 = 1.25(700 - 0.60(500)) = 500$ m

(b) What does Pete see for coordinates of E4? $x_4 = 1.25(688 - 0.60(813)) = 250$ m, $t_4 = 1.25(813 - 0.60(688)) = 500$ m

(c) Calculate the space-time interval between E3 and E4.

$\Delta t = 813 - 700 = 113$ m, $\Delta x = 688 - 500 = 188$ m

So $STI = \sqrt{113^2 - 188^2} = \sqrt{-22575} = (150\,\imath)$ m, an imaginary answer. What does this mean? Certainly the events can occur wherever they want to, but then some space-time intervals will be real, some imaginary, and some zero.

We will discuss this further in the next chapter, and describe time-like, space-like, and light-like space-time intervals

Example Ursula stands at the rear of a ship that she measures to be $L = 100$ m long with her friend Ulla at the front. They watch Pete fly by moving to the left at 0.60; Pete is at the front of his ship and Paul is at the rear. At $t = 0$ m Ursula measures two events, with the help of her friend Ulla. E1: At $x = 0$ m a spark is sent from the front of Pete's ship to the rear of Ursula's ship. At $x = L = 100$ m a spark is sent from the rear of Pete's ship to the front of Ursula's ship.

(a) According to Ursula, how long is her ship? Is this a proper length?

(b) Draw a space-time diagram as seen by Ursula, and on it mark the two events and appropriate world lines.

(c) Use the Lorentz transformation to find the x' and t' coordinates of the two events as measured by Pete.

(d) According to Ursula, how long is Pete's ship? Is this a proper length?

(e) According to Pete, how long is Ursula's ship?

(f) According to Pete, how long is his ship?

(g) Draw a space-time diagram as seen by Pete. On it mark the two events.

(h) Who sees the proper time between the two events, Pete or Ursula?

(i) Draw an event diagram (picture) of the events as seen by Pete.

(j) Ursula would say "The spark between Pete and me occurs simultaneously with the spark between Paul and Ulla." What would Pete say about these two events?

(k) According to Pete, simultaneous with the spark between Ulla and Paul, Ursula sends a spark. This is event E3. Find the coordinates of this event according to Pete.

(l) On the two space-time diagrams, mark E3.

(m) Using each space-time diagrams, determine at what fraction of the length of Pete's ship the char mark from the spark in E3 occurs. Do the two diagrams give the same result?

3.4 The Velocity Transformation

Ursula and Pete are watching a third rocket in which Ted is riding. Ursula measures the velocity of Pete, Pete measures the velocity of Ted. From these two velocities we would like to know what result Ursula would measure for the velocity of Ted.

Notation: We will use v for velocity, but must keep track of who is doing the measuring. A couple of notations are useful. We could add two subscripts, one for the object that is moving, and one for the observer. Or we could use a superscript, none for Ursula's measurement, and a prime, v' for Pete's measurement. To indicate the velocity of Ted as measured by Ursula we could write $v_{TU} = v_T$. For the velocity of Ted as measured by Pete we could use $v_{TP} = v'_T$.

If Ted is the only object being measured we could omit the T subscript entirely. Eventually we will look in 2D and 3D and will need another subscript for the component of the velocity.

The relation between velocities measured by Ursula and Pete is known by a variety of names: *Velocity Addition Formula, Law of Combination of Velocities*, or my choice, *Velocity Transformation.*

I'll begin with the non-relativistic Galilean Transformation.

3.4.1 Non-Relativistic Galilean Transformation of Velocities

Start with Equation 3.4, $x' = x - \beta t$. Here is a non-calculus derivation. An object moves in 1D from position x_1 to x_2 in some time interval Δt.

Write the coordinates of two different events as measured by Pete,

$$x'_2 = x_2 - \beta\, t_2 \qquad (3.15)$$

$$x'_1 = x_1 - \beta\, t_1 \qquad (3.16)$$

and subtract

$$\Delta x' = \Delta x - \beta\, \Delta t \qquad (3.17)$$

Now divide by the time interval,

$$\frac{\Delta x'}{\Delta t} = \frac{\Delta x}{\Delta t} - \beta \qquad (3.18)$$

or

$$v' = v - \beta \quad \mathrm{Non-Relativistic} \qquad (3.19)$$

Or we can start with Equation 3.4 and do the derivative from calculus

$$\frac{dx'}{dt} = \frac{dx}{dt} - \beta \qquad (3.20)$$

This gives the same result, Equation 3.19.

So if Ursula measures Pete fly past in an airplane at 90 m/s east and Pete measures a ball thrown at 5 m/s east, then Ursula knows that she would measure the ball moving at 95 m/s east. This works great as long as the speeds are small compared with the speed of light.

3.4.2 Vectors and Components

We have been discussing the position of an object in 2D and have given its coordinates, relative to an origin, as x and y. It is also possible to give the position in terms of a distance and an angle, r and θ. The relation between the two ways of presenting the data are shown in Figure 6.1, and in symbols are

$$r = \sqrt{x^2 + y^2} \qquad (3.21)$$

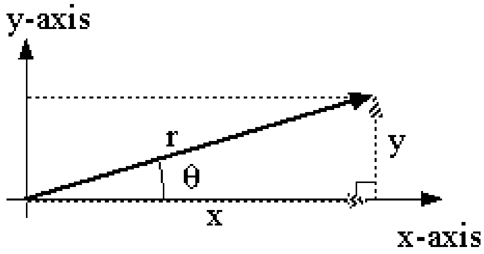


Figure 3.4: A two dimensional vector can be described either by components, x, y, or by length and angle, r, θ

$$\theta = tan^{-1}\frac{y}{x} \tag{3.22}$$

If we know the distance and the angle we can use trigonometry to get the components.

$$x = r\cos\theta \tag{3.23}$$

$$y = r\sin\theta \tag{3.24}$$

The position is an example of a vector. Similar expressions are true for the vector velocity, we can give its components, v_x, v_y or its speed and angle v, θ where

$$v = \sqrt{v_x^2 + v_y^2} \tag{3.25}$$

$$\theta = tan^{-1}\frac{v_y}{v_x} \tag{3.26}$$

and

$$v_x = v\cos\theta \tag{3.27}$$

$$v_y = v\sin\theta \tag{3.28}$$

3.4.3 Relativistic Velocity Transformation

To determine the x component of velocity of Ted we can use $v_{x,ave} = \Delta x/\Delta t$ and take small time intervals so the average is the instantaneous velocity. First use the Lorentz transformations to get expressions for the space and time intervals between two events.

$$\begin{aligned} x_2 &= \gamma(x_2' - \beta'\ t_2') \\ x_1 &= \gamma(x_1' - \beta'\ t_1') \\ \Delta x &= \gamma(\Delta x' - \beta'\ \Delta t') \end{aligned} \tag{3.29}$$

$$\begin{aligned} t_2 &= \gamma(t_2' - \beta'\ x_2') \\ t_1 &= \gamma(t_1' - \beta'\ x_1') \\ \Delta t &= \gamma(\Delta t' - \beta'\ \Delta x') \end{aligned} \tag{3.30}$$

Now the velocity in Ursula's reference frame can be written

$$\begin{aligned} v_x = \frac{\Delta x}{\Delta t} &= \frac{\gamma(\Delta x' - \beta'\ \Delta t')}{\gamma(\Delta t' - \beta'\ \Delta x')} \\ &= \frac{\Delta x'/\Delta t' - \beta'\ \Delta t'/\Delta t')}{\Delta t'/\Delta t' - \beta'\ \Delta x'/\Delta t')} \\ &= \frac{v_x' - \beta'}{1 - \beta' v_x'} \end{aligned} \tag{3.31}$$

We can similarly get the transformation for the transverse velocity components, using $\Delta y = \Delta y'$.

$$\begin{aligned} v_y = \frac{\Delta y}{\Delta t} &= \frac{\Delta y'}{\gamma(\Delta t' - \beta'\ \Delta x')} \\ &= \frac{\Delta y'/\Delta t'}{\gamma(\Delta t'/\Delta t' - \beta'\ \Delta x'/\Delta t')} \\ &= \frac{v_y'}{\gamma(1 - \beta' v_x')} \end{aligned} \tag{3.32}$$

Similarly

$$v_z = \frac{v_z'}{\gamma(1 - \beta' v_x')} \tag{3.33}$$

The velocity transformation for x-components involves only the relative motion (assumed to be in x direction) and the x-components of velocity. However the transverse velocities involve velocity components in both the transverse direction (y or z) and the velocities along the x direction, β and v_x. This will lead to curious results.

Differences between my Velocity Transformation and that of *Spacetime Physics*.

Equation L-13 in *Spacetime Physics* looks different than my Equation 3.31. This is because my formula uses only Pete's primed measurements on the right rather than mixing up Pete's primed and Ursula's unprimed measurements.

Notice that by using my notation there is no need for an inverse transformation, you just replace primed quantities with un-primed quantities and vice versa. This is summarized in Table 3.3.

I would suggest writing Excel formulas for the velocity transformations, `vxprime(beta, vx)` and `vyprime(beta, vx, vy)`.

3.5 Using Velocity Transformations

The key to using Equations 3.31 and 3.32 is to carefully determine which are the primed and which are the unprimed velocities, as well as to remember that $\beta' = -\beta$. The first set of examples will be for 1D motion.

Example Ursula measures Pete moving to the left at a speed of 0.60 and Ted moving to the right at a speed of 0.50. What is the velocity of Ted according to Pete?

$\beta = -0.60 \quad v_x = +0.50$

Then $v'_x = \frac{v_x - \beta}{1 - \beta v_x} = \frac{0.50 - (-0.60)}{1 - (0.50)(-0.60)} = \frac{1.1}{1.3} = 0.85$

Pete measures Ted moving to the right at 0.85.

Example Ursula measures Pete moving to the left at 0.60. Pete measures Ted moving to the right at 0.50. What does Ursula measure for the velocity of Ted?

$\beta = -0.60 \quad \beta' = +0.60 \quad v'_x = +0.50$

$v_x = \frac{v'_x - \beta'}{1 - \beta' v'_x} = \frac{0.50 - 0.60}{1 - (0.50)(0.60)} = -0.14$

Ursula sees Ted moving to the left at 0.14.

Example Ursula measures Pete moving (a) to the right at 0.95 (b) to the left at 0.95. Pete turns on a flashlight and measures the light moving to the right at 1.00. What does Ursula measure for the velocity of the light?

(a) $\beta = +0.95 \quad \beta' = -0.95 \quad v'_x = +1.00$

$v_x = \frac{v'_x - \beta'}{1 - \beta' v'_x} = \frac{1.00 - (-0.95)}{1 - (-0.95)(1.00)} = +1.00$

Ursula also sees the light moving to the right at 1.00.

(b) $\beta = -0.95 \quad \beta' = +0.95 \quad v'_x = +1.00$

$v_x = \frac{v'_x - \beta'}{1 - \beta' v'_x} = \frac{1.00 - 0.95}{1 - (0.95)(1.00)} = +1.00$

Table 3.3: Comparing Velocity Transformation Equations: v_z transformations are like the v_y transformations

	Equation	Text	Equation	Lindberg
x-component	L-13	$v_x = \frac{v'_x+\beta}{1+v'_x\beta}$	3.31	$v_x = \frac{v'_x-\beta'}{1-\beta' v'_x}$
Inverse x		$v'_x = \frac{v_x-\beta}{1-v_x\beta}$	3.31	$v'_x = \frac{v_x-\beta}{1-\beta v_x}$
v_y	Not given		3.32	$v_y = \frac{v'_y}{\gamma(1-\beta' v'_x)}$
v'_y	Not given		3.32	$v'_y = \frac{v_y}{\gamma(1-\beta v_x)}$

Ursula still sees the light moving to the right at 1.00.

These examples give us a hint that the speed of light in vacuum seems to be the fastest possible speed that any object can have.

Now consider 2D motion, with the transverse component being the y-axis. Ursula and Pete are still moving relative to each other along the x-axis, it is Ted that is moving in 2D.

Example Ursula measures Pete moving to the left at 0.60. Pete shines a flashlight and measures the beam moving at 1.00 at an angle of 37° to the x-axis. What is the velocity—size and angle (direction)—that Ursula measures for the light?

$\beta = -0.60 \quad \beta' = +0.60 \quad \gamma = 1.25$

$v'_x = +1.00\cos 37^\circ = +0.799$

$v'_y = +1.00\sin 37^\circ = +0.602$

$v_x = \frac{v'_x-\beta'}{1-\beta' v'_x} = \frac{0.799-0.600}{1-(0.799)(0.600)} = 0.382$

$v_y = \frac{v'_y}{\gamma(1-\beta' v'_x)} = \frac{0.602}{1.25(1-(0.60)(0.799)} = 0.925$

This means that $v = \sqrt{v_x^2+v_y^2} = \sqrt{0.382^2+0.925^2} = 1.00$

Light still moves at the speed of light in agreement with Einstein's Second Postulate. The angle with respect to the x-axis is

$\theta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{0.925}{0.382} = \tan^{-1} 2.42 = 67.6^\circ$

Example Ursula measures Pete moving to the right at 0.80 and Ted moving at 0.50 at 30° above the horizontal. What does Pete measure for Ted's velocity?

$\beta = +0.80 \quad \beta' = -0.80 \quad \gamma = 1.667$

$v_x = 0.50\cos 30^\circ = +0.433$

$v_y = +0.50\sin 30^\circ = +0.250$

$v'_x = \frac{v_x-\beta}{1-\beta v_x} = \frac{0.433-0.800}{1-(0.443)(0.800)} = -0.561$

$v'_y = \frac{v_y}{\gamma(1-\beta v_x)} = \frac{0.250}{1.667(1-(0.80)(0.433)} = 0.229$

This means that $v' = \sqrt{v'^2_x+v'^2_y} = \sqrt{(-0.561^2+0.229^2} = 0.606$

at an angle $\theta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{0.229}{-0.561} = 157.7°$

3.6 Doppler Shifts

Imagine Pete carrying with him a periodic source of pulses—this could be a tuning fork in the case of sound, or a laser in the case of light. Pete measures the period and frequency of the source, $T = 1/f$. Ursula sees Pete moving, and we will consider motion to be straight toward or straight away from Ursula. When Ursula receives the pulses, what is the received frequency?

We will define the following conventional picture. The source is on the left and the observer is on the right. Velocities to the right are considered to be positive, velocities to the left are considered to be negative.

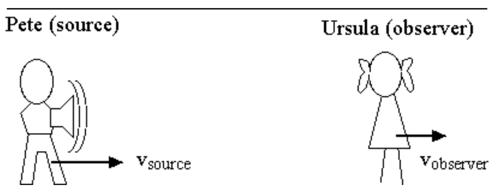


Figure 3.5: We use this for our conventional picture for the Doppler effect. The source is to the left and the observer to the right. Velocities to the right are positive, to the left are negative.

3.6.1 Dopper Equation for Sound

You may have heard of the Doppler effect for sound. When a sound source approaches you, you hear a higher frequency, when the sound source recedes, you hear a lower frequency. The equation is

$$f_{observed} = \frac{1 - v_{observer}/v_{sound}}{1 - v_{source}/v_{sound}} \quad \text{for sound}$$

where v_{sound} is the speed of the sound.

This formula works very well since sound moves in a medium, typically air. For a long time scientists believed in the ether (also spelled æther), a medium in which light was assumed to propagate. During the 19^{th} century this ether came to need stranger and stranger properties—it had to be rigid enough to let light propagate, but fluid enough to let the earth orbit the sun. The ether needed to be stationary in one special reference frame, that where Maxwell's equations were perfectly correct.

Michelson and Morley did an experiment in 1887 in Cleveland Ohio to determine the speed of the earth with respect to the ether. They found no result, one of the first pieces of evidence that led to the demise of the ether, and the rise of Special Relativity.

3.6.2 One-Dimensional Relativistic Doppler Effect

You can follow the following outline to derive the One-Dimensional Relativistic Doppler Effect. We will assume that Ursula is at the origin of her coordinate system and beginning at $t = 0$ sends pulses of light to the right separated by a period $T = 1/f_s = 2$ m. For Ursula, the time T is a proper time between her sending pulses, and therefore she measures the proper frequency, f_s, of the source.

At $t = 0$ Ursula measures Pete to be at $x_{init} = 5$ m and moving with speed $\beta = 0.60$ to the right.

(a) Start by drawing a space-time diagram using 1 small square as 1 m. Show the world lines for Ursula and Pete. Then draw the world lines for first three light pulses that Ursula sends.

Label the events that correspond to Pete receiving the pulses E0, E1, and E2. The coordinates for these events are $(t_0, x_0), (t_1, x_1), (t_2, x_2)$.

(b) Consider the light pulses. For the first pulse traveling at the speed of light, 1.00, $x_0 = t_0$. Write similar equations for x_1 and x_2, then generalize to the N^{th} pulse. Your answers should include T and N.

(c) Now consider Pete. He started at x_{init} and received the first pulse at t_0, so we can write $x_0 = x_{init} + \beta\, t_0$. Write similar expressions for the positions x_1 and x_2.

(d) Equating the two expressions for x_0 we can solve for t_0.

$$t_0 = x_{init} + \beta\, t_0 \qquad (3.34)$$
$$t_0 = \frac{x_{init}}{1-\beta}$$

and so

$$x_0 = \frac{x_{init}}{1-\beta} \qquad (3.35)$$

Repeat this process to get expressions for t_1, t_2, x_1, and x_2, then extrapolate for t_N and x_N. The expressions can contain β, x_{init}, N, and T.

(e) Now lets switch to Pete's viewpoint. Use Lorentz transformations to get the times t'_0 and t'_N, and subtract to get the interval between receiving pulse 0 and pulse N as measured by Pete. This difference is $N\,T'$.

(f) Use the fact that $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{(1-\beta)(1+\beta)}}$ to simplify the expression, and switch from period to frequency to get

$$f_{observed} = \left(\sqrt{\frac{1-\beta}{1+\beta}}\right) f_s \qquad (3.36)$$

This equation relates the source frequency f_s (where the light is generated), measured for a source at rest, to the frequency $f_{observed}$ measured by an observer moving with respect to the source. The source frequency, f_s, is the proper frequency measured by the source, and the observed frequency, $f_{observed}$, is the proper frequency measured by the observer. The relative velocity β in the equation is the speed of the source as measured by the observer, and is positive when the observer moves away from the source (or the source moves away from the receiver.).

When you use Equation **??**, it is easy to check that you used the correct sign for β. If the source and observer are approaching, $f_{observed} > f_s$ while if they are receding, $f_{observed} < f_s$.

If the source is moving, the same equation is valid and its use will be shown in the last example below.

Example Ursula carries a green laser that has a wavelength of 532 nm and a frequency $f_s = c/\lambda = \left(3\times10^8\, m/s\right)/\left(532\times10^{-9}\, m\right) = 5.64\times10^{14}$ Hz. Pete is moving away from Ursula at a speed of 0.10. What are the frequency and wavelength measured by Pete?

$$\begin{aligned} f_{observed} &= \left(\sqrt{\frac{1-.10}{1+.10}}\right)5.64\times10^{14} \\ &= 5.10\times10^{14}\, Hz \end{aligned}$$

$$\lambda_{observed} = c/f_{observed} = 588 \text{ nm}$$

Pete sees the light as being red. This is an example of red-shift. If Pete and Ursula were approaching the frequency would increase, the wavelength would decrease, and we would call it blue-shifted light.

Example If we observe a frequency 1/3 of the emitted frequency, (a) is the source approaching or receding and (b) what is the speed?

(a) Since the frequency is lower we must be receding.

(b) Use Equation 3.36 to get $\beta = +0.80$. The positive sign confirms the direction of motion.

Example If we observe a frequency three times (3×) the emitted frequency, (a) is the source approaching or receding and (b) what is the speed?

(a) Since frequency is increasing, we are approaching the source.

(b) Use Equation 3.36 to get $\beta = -0.80$. The negative sign confirms the direction of motion.

Example Jack sees Paris moving to the right at 0.60 and George moving to the right at 0.20, with George to the right of Paris. Paris is carrying a laser with wavelength of 700 nm. What is the wavelength measured by George?

Both are moving, and are moving closer. Therefore we know that the frequency observed by George is higher than the frequency emitted. We also know that $c = f\,\lambda$ and with wavelengths Equation 3.36 becomes

$$\lambda_{observed} = \left(\sqrt{\frac{1+\beta}{1-\beta}}\right)\lambda_s \qquad (3.37)$$

and if the observed frequency is higher, the observed wavelength is shorter.

We must use the velocity transformation equations to find β, the velocity of Paris seen by George.

$$\beta = \frac{0.60-0.20}{1-0.60(0.20)} = 0.455 \qquad (3.38)$$

Then we get the shorter wavelength by using a negative sign with β.

$$\begin{aligned} \lambda_{observed} &= \left(\sqrt{\frac{1+\beta}{1-\beta}}\right) \\ \lambda_s &= \left(\sqrt{\frac{1+(-0.455)}{1-(-0.455)}}\right)700 \\ &= 428 \text{ nm}. \end{aligned} \qquad (3.39)$$

3.6.3 Redshifts

If we know the frequency of the source in its reference frame, we can use the Doppler Effect to find speeds of approach or recession. Let's apply this to stars. It is very difficult to bring a known laser to even the nearest star, so how can we know the source frequency?

Fortunately nature gives us this information in terms of the emission spectra (and absorption spectra) of elements. For example, prominent lines emitted by Hydrogen (the Balmer series) are red, Hα, 656 nm, blue green, Hβ, 486 nm, and two violet lines at 434 nm, Hγ, and 410 nm, Hδ. The relative spacing of the wavelengths identifies the source as hydrogen, even if the wavelengths are Doppler shifted far away from the visible.

Astronomers define the red shift as

$$z_{redshift} = \frac{\lambda_{observed} - \lambda_{emitted}}{\lambda_{emitted}} = \frac{f_{emitted} - f_{observed}}{f_{observed}} \quad (3.40)$$

Figure 3.6 shows the visible Balmer lines as vertical bars, and the observed spectrum of quasar 3C 273.[3] The peaks in the observed spectrum are shifted toward longer wavelengths as are shown by the arrows. The computed red shift for this quasar is 0.158 and that translates to a speed of 0.146. If we look closely at an active galaxy such as M87, the nucleus will be rotating at a high speed. The overall galaxy may be red shifted by some average amount, but the part of the galaxy approaching us will be blue shifted relative to the average, while the part of the galaxy receding from us will be red shifted relative to the average. This is shown in Figure 3.7.

[3]This is one of the nearest quasars, discovered in 1959, and is now known to be the active galactic nuclei surrounding a massive black hole. For more information see `http://en.wikipedia.org/wiki/3C_273`

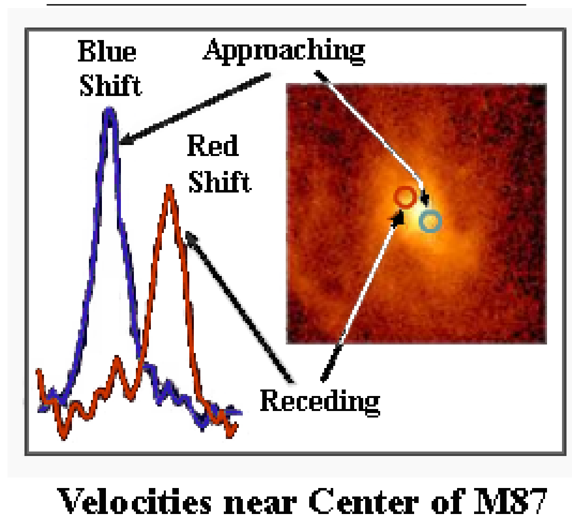


Figure 3.7: Red and blue shifts at the center of a rotating galactic core. These are shifts relative to the overall red- or blue-shift due to the overall motion of the galaxy.

Red shifts are also caused by the expansion of the universe and by strong gravitational fields (General Relativity), and these become important in the interpretation of some red-shift data. We will only consider red shifts due to Doppler effects.

Most stars and galaxies in the universe are red-shifted, but some nearby galaxies like Andromeda are blue-shifted. Andromeda has a velocity of -0.001 and a z-shift of -

0.001.

Example Find the red shift for a star (a) approaching at 0.40 (b) receding at 0.90.

(a) $\beta = -0.40$.

You can show that

$$z = \frac{\sqrt{1+\beta} - \sqrt{1-\beta}}{\sqrt{1-\beta}} = -0.34$$

(b) $\beta = +0.90$ and using the equation we get $z = 3.36$.

The largest red-shifts corresponding to quasars have $z = 7$.

3.7 Approximations

Relativistic effects are very small if we are moving at "low" speed, $\beta << 1$. However "low" relativistically is still very fast for us terrestrial observers. When we move slowly we can make approximations.

Probably the most common approximation is the *Binomial Approximation* for $(1+x)^n$ when $x << 1$.

$$(1+x)^n \approx 1 + n\,x + \frac{n(n-1)}{2}x^2 + \cdots \quad (3.41)$$

In calculus you can prove this using the Taylor series, but for now just consider one example with $x = 0.01$ and $n = 3$:

$$\begin{aligned}(1+x)^3 &= 1 + 3x + 3x^2 + x^3 \\ &= 1 + 3(0.01) + 3(0.01)^2 + 0.01^3 \\ &= 1 + 0.03 + 0.0003 + 0.000001 \\ &\approx 1.030\end{aligned}$$

Clearly the higher order terms in x contribute very little to the total and can be ignored. We are making a first-order approximation to the binomial.

For small values of β the binomial expansion of γ is

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\beta^2}} \\ &= (1-\beta^2)^{-1/2} \\ &\approx 1 - (\frac{-1}{2})\beta^2 \\ &= 1 + \beta^2/2 \qquad (3.42)\end{aligned}$$

Try $\beta = 0.3$. The exact value is $\gamma = 1.0483$ and the approximation is 1.0450, quite close. For slower speeds the result is even better. Try $\beta = 0.1$.

Example Ursula sees a proper time interval of 100.00000 s. She sees Pete moving at 1380 m/s (Speed of space shuttle.) What is the time interval according to Pete?

First lets get the dimensionless speed and γ.

$\beta = 1380/3 \times 10^8 = 4.6 \times 10^{-6}$ $\gamma \approx 1 + (4.6 \times 10^{-6})^2/2 = 1 + 1.058 \times 10^{-11}$

If I try to sum these on my (cheap) calculator I just get 1, so I keep the terms separate. The improper time interval measured by Pete is 1.058 ns longer than the proper interval measured by Ursula.

3.8 Summary

It is often convenient to describe position and time of events, rather than interval between two events. Lorentz Transformations allow the conversion of values from one inertial observer to another.

$$\begin{aligned} t &= \gamma(t' - \beta'\, x') \\ x &= \gamma(x' - \beta'\, t') \\ y &= y' \quad z = z' \end{aligned} \tag{3.43}$$

From the Lorentz Transformations we can get the Velocity Transformations. The velocity transformations show that nothing can travel faster than the speed of light.

$$\begin{aligned} v_x &= \frac{v'_x - \beta'}{1 - \beta' v'_x} \\ v_{y,z} &= \frac{v'_{y,z}}{\gamma(1 - \beta' v'_x)} \end{aligned} \tag{3.44}$$

Also we can find the Relativistic Doppler Effect Formula that explains frequency changes due to motion along the line joining source and observer.

$$f' = \left(\sqrt{\frac{1-\beta}{1+\beta}} \right) f \tag{3.45}$$

In astrophysics, overall motion of a galaxy relative to us, the observers, results in a Doppler Shift, toward blue for approaching galaxies (rare) or toward red for receding galaxies. The red shift z is frequently used to describe the motion. Other sources of red shift also exist, but are not discussed here.

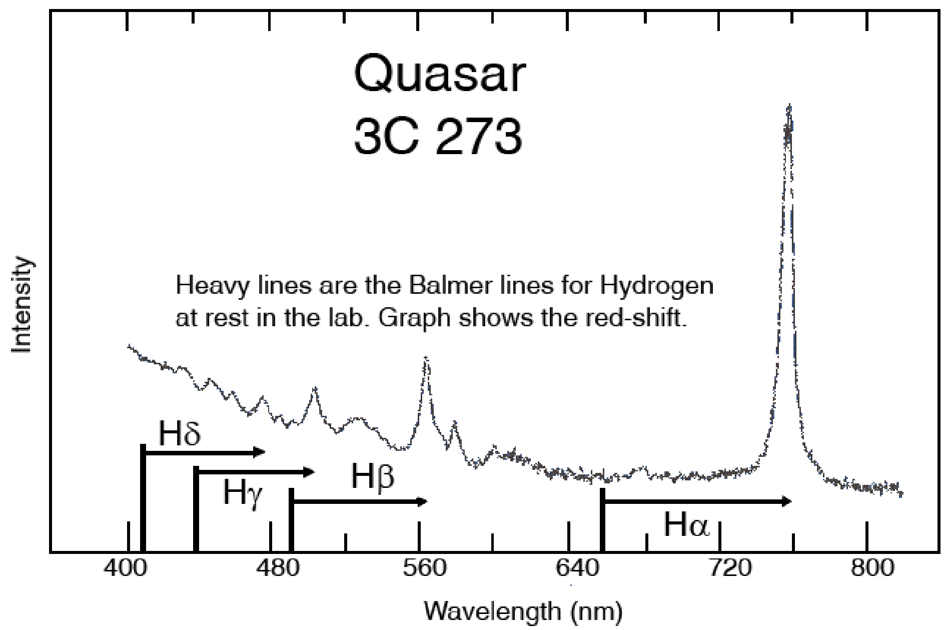


Figure 3.6: Red shift of the Hydrogen Balmer series for quasar 3C 273. Vertical bars are the locations of lines for hydrogen in the lab. Shifts in the observed spectrum are shown by the arrows. This quasar has a red shift of 0.158, and a speed of 0.146.

Chapter 4

Paradoxes and Their Resolution, Moving Along a Curved Worldline

4.1 Introduction

In this chapter we will examine a whole bunch of problems and paradoxes that will help us understand special relativity kinematics. At the end we will discuss time along a general world-line, one that allows an observer to accelerate rather than move at constant velocity.

4.2 The equations

Here are the equations that we have used in the first three chapters. I have used my versions of the Lorentz Transformations, Equations 4.6, and the Velocity Transformations, Equations 4.7, that have only primed quantities on the right hand side, and unprimed on the left.

The two observers may be considered Ursula (unprimed) and Pete (primed.) Pete is moving with a constant velocity β with respect to Ursula. The primed and unprimed axes are parallel, motion is along the $x - x'$ axes and the origins coincide at time 0. Both are inertial reference frames to the limit of accuracy of our measurements.

$$\beta' = -\beta \tag{4.1}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{4.2}$$

$$\Delta t^{improper} = \gamma\ \Delta t^{proper} \tag{4.3}$$

$$L^{improper} = L/\gamma \tag{4.4}$$

$$STI^2 = \Delta t^2 - \Delta x^2 \tag{4.5}$$

$$\begin{aligned} t &= \gamma\left(t' - \beta'\ x'\right) \\ x &= \gamma\left(x' - \beta'\ t'\right) \\ y &= y' \quad z = z' \end{aligned} \tag{4.6}$$

$$\begin{aligned} v_x &= \frac{v'_x - \beta'}{1-\beta' v'_x} \\ v_y &= \frac{v'_y}{\gamma(1-\beta' v'_x)} \\ v_z &= \frac{v'_z}{\gamma(1-\beta' v'_x)} \end{aligned} \tag{4.7}$$

$$f' = \left(\sqrt{\frac{1-\beta}{1+\beta}}\right) f \tag{4.8}$$

4.3 Seeing versus Measuring

Spacetime Physics problem 3-15.

You are located in Rochester on a flat earth. A rocket is heading toward you, with velocity β. As it passes the Golden Gate bridge in San Francisco it sends out a pulse of light. It sends a second pulse of light when it passes under the Gateway Arch in St. Louis. Based on unintelligent observation, what would you see for the velocity of the rocket?

An event diagram is shown below. Assume that for you on the earth, the second pulse occurs a time Δt after the first.

(a) What is the distance between San Francisco and Saint Louis in terms of Δt?

(b) At the time of the second pulse, where is the first pulse located, in terms of Δt?

(c) What is the space separation of the two pulses? What is the time separation between the two pulses?

(d) A naïve calculation of the rocket speed would be to take the known distance between the two cities and divide by the time interval between when you see the two pulses. What doe this give for v^{seen}?

(e) Suppose the rocket were traveling at $\beta = 0.5$. What would be v^{seen}?

(f) Suppose that $v^{seen} = 4$. What is the true speed of the rocket?

(g) Suppose that your friend were in San Francisco. What would she see for the velocity of the rocket in each of the last two cases? Hint: you can either reanalyze the situation, or use the equation already derived and a little cleverness to get the new equation.

(h) Resolve the paradox. Why do we see the rocket moving faster than light speed?

4.4 Seeing the Length of a Rod

Pete rides on the primed rocket carrying a stick that is parallel to the x'-axis and of length $L' = 100$ m. His friend Paris is to his right and proposes the following method by which Ursula can determine the length of the stick. Ursula is at the origin of her coordinate system, and she sees Pete and Paris move to the right at $\beta = 0.60$.

"At time $t' = 10$ m Pete and I will stand at either end of the stick (Pete will be at $x' = 50$ m) and send light pulses towards Ursula. Ursula will measure the time between the pulses and knowing that the dimensionless speed of light is 1.00 can say that the length of the rod equals the time difference between receiving pulses."

(a) Without any calculation, can you see the flaw in the logic? What is it?

(b) Write the coordinates of the two events (Pete sends flash, Paris sends flash) as measured by Pete.

(c) Do a Lorentz transformation to find the coordinates of the events as measured by Ursula.

(d) At what times will the two pulses arrive at Ursula who is standing at the origin

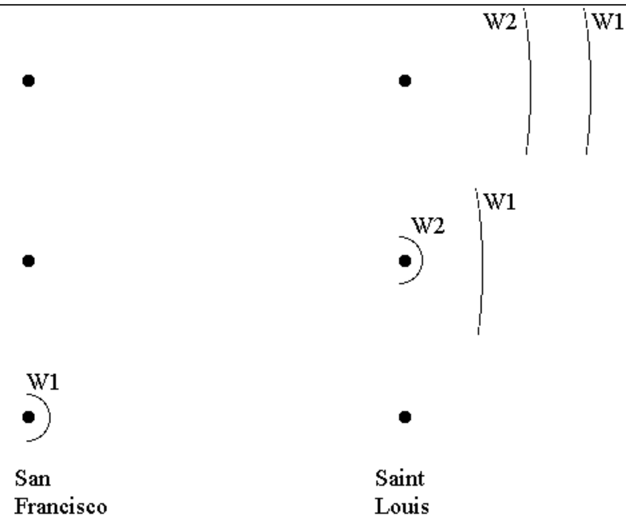


Figure 4.1: Seeing versus Measuring: Event diagram for a rocket heading eastward on a flat earth.

of her coordinate system?

(e) Using Paris's method, what would be the apparent length of the stick as seen by Ursula?

(f) What is the actual length of the stick as measured by the intelligent observer Ursula?

4.5 Scissors Paradox

Spacetime Physics problem 3-14 a. We simulate a closing scissors by having one horizontal fixed rod, and a second rod tilted at angle θ to the horizontal and moving vertically downwards with speed β. Consider the point where the two rods cross. This point moves to the right at some speed v_A. Determine v_A in terms of β and θ. Find the conditions under which this speed is larger than 1 (i.e. faster than light speed.)

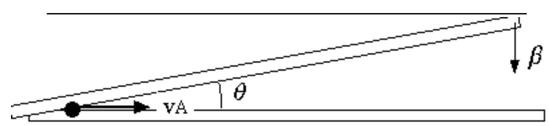


Figure 4.2: Scissors Paradox. The horizontal rod is at rest, top rod is tilted and falling. The point of intersection of the rods can move faster than the speed of light.

On the figure, draw the rod after it has fallen for a short amount of time Δt. Write expressions for distances and then use trigonometry.

In Section 4.3 we had a situation where the observed speed of an object exceeded 1, however this was the result of NOT using intelligent observers. In the Scissors Paradox we do not have that problem—the intersection point can travel at any speed up to infinity. Does this violate Einstein's postulates. If not, why not?

4.6 What If I Could Move Faster Than Light?

Spacetime Physics Sample problem L-2. Here is another case of proof by *reductio ad absurdum*.

Assume that an object can have velocities both less than and greater than the speed of light in vacuum. What does this lead to? We will see that it leads to the absurd conclusion that you could be killed before you are born. Therefore we must reject the initial assumption and say rather that objects can only have speeds $0 < v < c$.

Event E1: At a time -4 years on the planet Klingon, the Peace Treaty of Shalimar was signed between Klingon and the Federation. Immediately upon signing the treaty, a Federation ship set out on impulse power at a speed of 0.60 to bring the treaty to Earth.

The murderous Klingons immediately started work on a super drive that would allow them to move faster than the speed of light. They perfected their drive and . . .

Event E2: . . . at time 0 the Klingons launched a ship at speed 3.00 to overtake the Federation ship and destroy it. **Event 3** is the destruction of the Federation ship.

(a) Determine the position and time coordinates of each of the events as measured by an observer on Klingon.

(b) Draw a space-time diagram from the vantage point of the Klingon home planet.

(c) Use Lorentz transformations to find the coordinates of each of the events as measured by the Federation crew.

(d) Draw a space-time diagram from the vantage point of the Federation ship.

(e) For observers on the planet Klingon, the Federation ship was destroyed after it was launched. For observers on the Federation ship, what is the order of events?

By working through this exercise you should come to the conclusion of *Spacetime Physics*,

> "What have we here? A confusion of cause and effect, a confusion that cannot be straightened out as long as we assume that the Super—or any other material object—travels faster than the speed of light in a vacuum.
>
> "Why does no signal and no object travel faster than light in a vacuum? Because if either signal or object did so, the entire network of cause and effect would be destroyed, and science as we know it would not be possible."

4.7 Change of Measured Orientation of Objects

Spacetime Physics Problem L-6a.

Pete rides a rocket carrying a stick of length L' that is tilted at an angle θ' from the x'-axis. Ursula in the lab sees Pete move to the right at β.

(a) What are the x' and y' components of the stick's ends?

(b) What are the x and y components of the stick's ends as measured by Ursula—assume an intelligent observer?

(c) What is the length of the stick as measured by Ursula?

(d) What is the angle the stick makes with the horizontal as measured by Ursula?

(e) Evaluate your expressions for (c) and (d) for the case of $L' = 100$ m, $\theta' = 30°$, and $\beta = 0.60$.

4.8 The Headlight Effect

Spacetime Physics Problem L-9.

As usual, Pete is riding a rocket moving to the right at β as measured by Ursula. Pete sends a pulse of light oriented at an angle θ' to the x'-axis.

Show that for Ursula the light makes an angle θ with

$$\cos\theta = \frac{\cos\theta' - \beta'}{1 - \beta' \cos\theta'} \tag{4.9}$$

This should be a really easy answer!

Now consider the light emerging from a light bulb that Pete carries. According to Pete the light is evenly spread in all directions, with half of the light emerging with angles between $-\pi/2$ and $\pi/2$. We say the half-angle is $\pi/2$.

Find the expression for the half-angle in Ursula's reference frame.

Evaluate in the case of $\beta' = -0.95$.

The headlight effect says that a moving light emits light that is concentrated in a small cone in the forward direction.

4.9 The Rising Stick.

Spacetime Physics Problem L-10.

A stick of proper length $L = 20$ m is parallel to Ursulas's x-axis and is rising at a dimensionless speed $v_y = 0.5$. Pete is seen to move to the right at $\beta = 0.6$. These numbers are measured by Ursula.

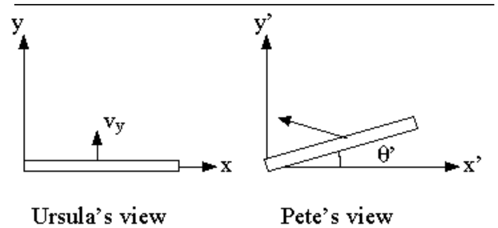


Figure 4.3: The Rising Stick: Observers move along x-axis, Stick is rising along y-axis for Ursula.

(a) Explain without doing calculations why the stick should be tilted according to Pete.

(b) According to Ursula, the left end of the stick is at $x = 0$, $y = 0$ m at $t = 0$ m. This is event E1: left end crossing the x-axis. What are the coordinates for event

E2 (x, y and t), the right end crossing the axis according to Ursula?

(c) What are the x'- and y'-components of the stick's velocity according to Pete?

(d) What are the coordinates (x', y' and t') of the two events according to Pete?

(e) Now consider a third event, the location of the left end of the stick at the same time as event E2. What are the x' and y' values for this event?

(f) From the results get the horizontal length of the stick as measured by Pete. Does this agree with the length contraction formula?

(g) What is the angle from the horizontal, θ', that Pete measures for the stick?

(h) If you have done this using numbers, go back and do it with symbols (all parts), ending up with $\tan\theta' = \gamma\beta v_y$.

Application: Consider problem L-11 in *Spacetime Physics*. Here is Ursula's view. A stick that is exactly 1 m (proper length) long moves horizontally at β. A plate with a hole of diameter exactly 1 m rises vertically at v_y. The stick is Lorentz contracted and therefore should fall through the hole. But consider the viewpoint of Pete riding on the stick. For him it is the hole that is Lorentz contracted and the stick should not fall through the hole. How can this paradox be resolved?

4.10 Paradox of the Identically Accelerated Twins

Problem L-13 in *Spacetime Physics* Twins Andrew and Brad own identical spaceships. Andrew's ship is on the left and Brad's is on the right separated by some distance. At time 0 in the earth frame they are 20 years old and accelerate to the right until they run out of fuel. According to Mom and Dad on earth, the twins run out of fuel at exactly the same time, and then have the same speed v_{final}, and they are separated by exactly the same distance as they were at the start.

Andrew and Brad compare notes and agree that they had the same history of accelerations, but find that Brad is older than Andrew. How can this happen?

We will look at a simpler situation where the acceleration occurs instantaneously by having the twins jump from one ship to another. Here is the trip as measured by Mom and Dad, with several events given.

Table 4.1: Andrew and Brad take a trip. Event coordinates measured by Mom and Dad on earth in units of days.

Event	x	t
E0 Andrew on ship A	0	0
E4 Brad on ship D	10	0
E1 Andrew jumps $A \rightarrow B$	1	5
E5 Brad jumps $D \rightarrow E$	11	5
E2 Andrew jumps $B \rightarrow C$	4	10
E6 Brad jumps $E \rightarrow F$	14	10
E3 Andrew on C, fuel gone	8.5	15
E7 Brad on F, fuel gone	18.5	15

This means that at $t = 0$ days Andrew and Brad stepped on spaceships A and D traveling at some speed to the right. After 5 days they each jump onto faster spaceships B and E for another 5 days. Finally they jump on spaceships C and F that are going even faster, until those ships run out of fuel.

(a) Find the velocities of each of the spaceships A through F.

(b) How much does each twin age during the trip? This is NOT 15 days!

(c) How far apart are the twins at the start and at the end of the trip?

(d) Draw a space-time diagram for the events as seen by Mom and Dad. Label each event on the diagram. Draw the world line for each twin by joining the points.

(e) Now consider the events as seen by the pilots of ships C and F. Find the coordinates of the 8 events by these pilots.

(f) Draw a space-time diagram as seen by these pilots. Draw the world line for each twin.

(g) In order to measure the separation of the twins, you need to measure two events that occur at the same time. Extend the world lines so that you can find the separation at the earliest and latest times that you found by using the Lorentz transformations. This is as measured by the pilots of the last ships. What are the separations at these two times.

(h) If the twins were of identical age at events E0 and E4, by how much have they aged when you compare their ages at the end of the trips, at the same times.

The seemingly absurd results that occur in problems like this can be verified experimentally. In this problem, as in so many other paradoxes, the resolution to the paradox lies in remembering that events that are simultaneous in one reference frame are no simultaneous in a different reference frame moving with respect to the first. As strange as it seems, we must embrace this experimental reality in order to solve problems in special relativity.

4.11 The Twin Paradox

Spacetime Physics, Chapter 4 discusses this in detail. Read it carefully to complement what I will discuss.

The most famous of Special Relativity Paradoxes is the Twin Paradox, in part because it was at the center of the 1963 novel *La planète des singes* by Pierre Boulle, more familiar to us as *Planet of the Apes*, source for innumerable movies beginning in 1968.

As Wikipedia summarizes, “Ulysse begins by explaining that he was friends with Professor Antelle, a genius scientist on Earth, who invented a sophisticated spaceship which could travel at nearly the speed of light. Ulysse, the professor, and a physician named Levain fly off in this ship to explore outer space. They travel to the nearest star system that the professor theorized might be capable of life, the red sun *Betelgeuse*, which would take them about 350 years to reach. Due to time dilation, however, the trip only seems two years long to the travelers."[1]

We will consider a famous pair of twins, Danny DiVito and Arnold Schwarzenegger.[2] Danny stays on earth while Arnold blasts off making a trip to *Proxima Centauri*, 4.2 years away, traveling at $\beta = 0.9$ for which $\gamma = 2.29$. Danny says that the trip takes $t_D = 4.2/0.9 = 4.67$ years, while Arnold, the proper observer, says the trip takes $4.67/2.29 = 2.03$ years.

Upon reaching *Proxima Centauri*, Arnold immediately returns at the same speed. Danny says that the round trip takes 9.33 years while Arnold claims that it takes 4.06 years. By this analysis Arnold is younger than Danny at the end of the trip.

Well that is strange enough, but Arnold makes the following claim. "In my reference frame I am at rest, and it is Danny and the rest of the universe that moves first at $\beta' = -0.9$, then at $\beta' = 0.9$. I see the improper distance travelled during the first half of the trip by Danny and the stars to be $L_A = 4.2/2.29 = 1.83$ years and this takes $1.83/0.9 = 2.03$ years. Danny measures the proper time of $2.03/2.29 = 0.89$ years. I claim the round trip will take 4.06 years but Danny will measure 1.78 years, and Danny is younger than me when we meet again."

There is a problem—only one of the two can actually be older. But who is correct? Can you see the resolution to this problem, i.e can you tell whether Danny or Arnold is older?

The resolution lies in deciding who sees an acceleration. From riding in cars, you know that you can sense an acceleration. Danny never feels an acceleration, while Arnold does feel an acceleration when he switches direction at *Proxima Centauri*. Therefore Danny's analysis is correct, and Arnold is younger than Danny when they meet again.

Now imagine a slightly more complicated situation, Danny and Arnold both board rockets and make round trips traveling at different speeds so as to return to earth at the same time. Both have experienced accelerations. Who is older and who is younger?

We will examine the Twin paradox closely in order to answer this question. Notice that Special Relativity is perfectly well equipped to deal with situations of accelerations. We start with a simple situation.

Example Consider the twin paradox in this way, using space-time diagrams. The twins will be called Ursula and Pete. Ursula stays on earth the entire time, Pete makes a trip to Proxima Centauri, located 4 light years away from earth, and back. Heading out he rides on Spaceship Alpha that travels at $\beta = 0.80$, and on the return he rides on Spaceship Omega that travels at $\beta = -0.80$. All the numbers mentioned so far are measured by Ursula.

Pete departs on July 4 2006. He and Ursula agree that they will each send a

[1] http://en.wikipedia.org/wiki/Planet_of_the_Apes Boulle underestimated the distance to *Betelgeuse* which is approximately 427 years. In either case, the rocket must travel at $\beta > 0.99998$.

[2] The movie *Twins*, 1988, "Only their mother can tell them apart."

light pulse to the other every July 4.

(a) According to Ursula, how long will it take for Pete to reach Proxima Centauri?

(b) Use graph paper and draw the space-time diagram as seen by Ursula. On it you should have world lines for Ursula and Pete (i.e. both spaceships), and for the pulses sent by Ursula. Mark the events when Pete receives the pulses from Ursula. How many pulses has Pete received, including the final one when he returns to earth? This is the age of Ursula when they meet again.

(c) Now compute quantities as seen by the Captain Ahab of Alpha. What is the velocity of Spaceship Omega as seen by Ahab? How long is Pete on Spaceship Alpha?

(d) Draw the space-time diagram for Ahab, including world lines for both Spaceships and Ursula and the pulses sent by Pete while he is on Alpha. Mark the events when Ursula receives a pulse from Pete. How many pulses has Ursula received on the outbound leg?

(e) By symmetry, on the inbound leg Ursula will receive the same number of pulses. What is the total number of pulses received by Ursula for the round trip? This is the age of Pete when they meet again.

4.12 Position and Time Not Absolute

Spacetime Physics, Chapter 5 will next be discussed.

Consider an event, you at the start of this class. The event has a definite time and a definite location. However the numerical values of the time and location are not unique, but vary depending on the observer.

Consider two cities, Rochester and Cleveland, and two observers, Alan and Beth. Alan and Beth start in Cleveland and agree to call the position there '0'. They want to know the distance from Cleveland to Rochester. Alan drives along I90 passing by Buffalo and when he gets to Rochester his odometer reads 261 miles. Beth visits friends at UPitt, Penn State, and Cornell on her way and her odometer reads 449 miles.

Both Alan and Beth make valid readings. Both agree that Cleveland and Rochester have definite locations, however they get different distances between the two cities.

In the same way we have established the rules that show that the time interval between two events is not unique, but depends on the observer.

For relativity we have an invariant spacetime interval, $STI^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$.

On a space-time drawing, two events, E1 and E2, that occur at the same location occur on a vertical line, two events, E1 and E3, that occur at the same time— i.e. simultaneously— occur on a horizontal line.

If we switch to the frame of a different observer events E1 and E2 will no longer occur at the same location. Events E1 and E3 will no longer occur at the same time, i.e. no longer be simultaneous. The STI between any two events will, however, be the same for all observers.

Let's return to our parable of the surveyors from Chapter 1. Different observers found different coordinates for parts of the city, however the distance between buildings was invariant, i.e. the same regardless of observer. In Section 1 of Chapter 3 we described a method to produce a map of the city given only distances between buildings. How does this translate to the case of special relativity?

The invariant quantity for the surveyor's was the distance, $\Delta r = \sqrt{\Delta x^2 + \Delta y^2}$ This is the equation of a circle. The arrows shown on the left of Figure 4.4 all are the same distance from the center. In the case of special relativity the invariant is the STI, in one dimension

$$STI^2 = \Delta t^2 - \Delta x^2 \tag{4.10}$$

This is the equation of an hyperbola, and assuming that $\Delta t > \Delta x$ its principal axis is time, shown on the right of Figure 4.4.

Recall that in the surveyor's parable, the two surveyors had different direction for North. However both would agree that the same circle represents points a distance 5 apart. In the same way in special relativity, different observers will measure different time and space intervals, but all agree that the hyperbola represents events separated by a STI of 5. Unlike the case of the surveyors, the arrows showing STI appear of different length, but the computed STI is exactly the same.

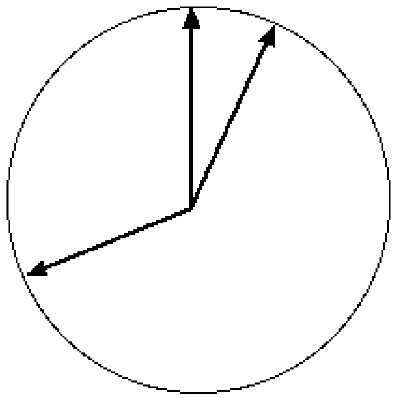
Here is another way to view the hyperbola. Imagine that a rocket, Alpha, moves to the right. Alpha carries a light flasher and a mirror, as shown in Spacetime Physics Figure 5-2. Consider two events: E1 = flash produced, it bounces off mirror and E2 = flash received back at location of flasher. Alan, in the rocket, can draw an hyperbola showing an STI between the events. For Alan $\Delta x = 0$, so the vertical arrow connects the two events as measured by him.

Consider Beth in the lab. She can measure the space and time intervals, compute the STI, and will get the same result as Alan. She will draw an identical hyperbola as did Alan to show points at the same STI. For her $\Delta x > 0$, so the arrow pointing up and to the right connects the two events as measured by her.

Finally consider Sam, riding in a super-rocket moving faster than Alpha to the right. Sam will measure time and space intervals between the events and get the same STI and therefore draw the same hyperbola as Alan and Beth. Sam will measure a space interval that is negative, and the arrow pointing up and to the left connects the events as measured by Sam.

So different observers will draw exactly the same hyperbola, the invariant hyperbola, for a given STI.

We can check to see that the hyperbola and



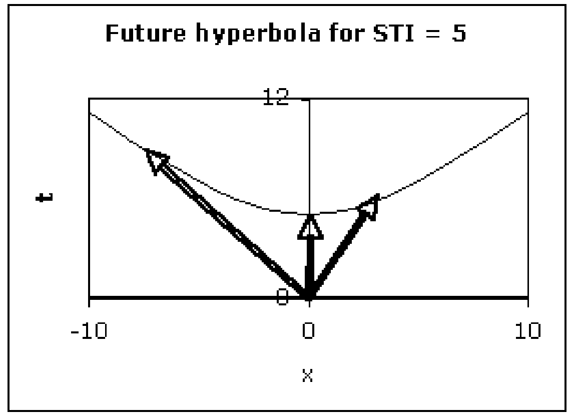


Figure 4.4: On left, surveyor's parable. Arrows connect points at the same distance from the center. On the right, an hyperbola representing something happening in the future. The arrows connect events at the origin with different events on the hyperbola. All arrows have the same value for the space-time interval, STI.

the Lorentz transformations yield the same results. Consider Alan who sees ($\Delta t = 5, \Delta x = 0$). Alan draws the hyperbola in Figure 4.4, with the vertical arrow connecting the two events. The coordinates of the second event as seen by Beth are (let's pick) ($\Delta t = 7.81, \Delta x = 6.00$) m.

What is Beth's velocity, according to Alan? Beth must be moving to the left, and with velocity $\beta = -\Delta x/\Delta t = -6.00/7.81 = -0.768$.

Now we can check the Lorentz transformations to see if they give the same results. E1 has coordinates (0,0) for both Alan and Beth. E2 has coordinates (5,0) m as measured by Alan. We compute $\gamma = 1.562$, then

$$\begin{aligned} t^{Beth} &= \gamma(t^{Alan} - \beta x^{Alan}) \\ &= 1.562(5 - (-0.768)0) \\ &= 7.81\, m \\ x^{Beth} &= \gamma(x^{Alan} - \beta t^{Alan}) \\ &= 1.562(0 - (-0.768)5) \\ &= 6.00\, m \end{aligned} \tag{4.11}$$

So our results are consistent.

4.13 General Worldline

Previously we drew worldlines for objects moving with constant velocities. But life is not really like that, velocities will increase and decrease as acceleration occurs. What is the most general form of a worldline?

Recall that on a space-time diagram, with time vertical, position horizontal, the slope of a straight line equals the inverse of the velocity:

$$Slope = \Delta t/\Delta x = 1/\beta \tag{4.12}$$

and that the speed of an object must not exceed the speed of light, $\beta \leq 1$. This means that the slope, $1/\beta \geq 1$.

Frequently we use the same scale for time and position, in which case the above statements tell us that the slope of a worldline must have an angle with the *time* axis

$$-\pi/4 = -45^\circ \leq \theta \leq \pi/4 = 45^\circ \tag{4.13}$$

This is shown in Figure 4.5, where the worldline on the left is possible. At the end of the worldline I have drawn two lines at 45° that delineate possible futures of the worldline. Regions shaded are not allowed for the particle. The two dotted lines define the *light cone*, and the worldline (past or future) can only exist inside the lightcone. You should be able to explain why the diagram on the right is impossible.

4.14 Time Kept By Traveller on a World Line

Let's say that Figure 4.6 shows a curved worldline for Beth as measured by Alan, and two events, E1 and E2. Each of them carries a clock (either a real clock or a biological clock). The space interval is the same for both Alan and Beth, $\Delta x = 0$, and by our ideas of proper time interval, both read a proper time interval between the events. We shall see that these two proper times are different, however.

The proper time interval that Alan reads is the time interval read on a clock held by him, the proper time that Beth reads is the proper time read on a different clock held by her. How do the two time intervals compare?

To answer this, break the curved path up into short pieces where the velocity is almost constant. E1 and E3 are two events that fit this description. Only Beth will see the proper time for the interval, and

$$\Delta t_{13}^{Alan} = \gamma \Delta t_{13}^{Beth} \tag{4.14}$$

Alternately we could look at the space-time interval, recalling that $\Delta x_{13}^{Beth} = 0$,

$$(\Delta t_{13}^{Beth})^2 = (\Delta t_{13}^{Alan})^2 - (\Delta x_{13}^{Alan})^2 \tag{4.15}$$

If we add all the proper time intervals for Beth between E1 and E2 we will get a shorter time interval for Beth's clock than Alan would measure for his clock. Spacetime Physics refers to this as the **Principle of Maximal Aging**: Between two events an observer that travels on a straight line on a space-time diagram will measure the largest time interval.

Let's look at a specific example.

Example From Alan's viewpoint, he is at rest, Beth undertakes the path described by the following table of data, and Carl moves to the right at a constant speed of 0.30.

(a) Draw a space-time diagram from Alan's viewpoint. Label the world lines of Alan, Beth, and Carl (assume he starts at $x = 0, t = 0$).

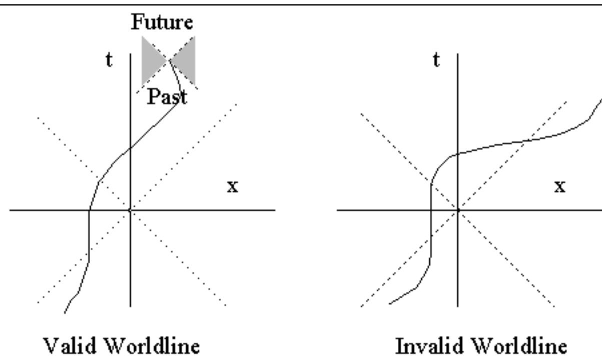


Figure 4.5: Space-time diagram showing a worldline for a particle. In general the worldline must be within 45° of the t-axis. At the top end of the line the light cone is drawn showing possible past path and allowed future path of the particle. The worldline on the right is impossible—you should be able to explain why.

Table 4.2: Coordinates of Beth, measured by Alan

t (m)	x (m)
0	0
1	0
2	0.2
3	0.8
4	1.4
5	1.6
6	1.4
7	0.8
8	0.2
9	0
10	0

(b) For each small interval, compute $\Delta t, \Delta x$ as measured by Alan, and compute the proper time interval measured by Beth.

(c) At the end Alan claims that 10 m of time have passed. What does Beth claim? Does this agree with the Principle of Maximal Aging?

(d) Now consider Carl's measurements for the world lines of Alan and Beth. Do Lorentz transformations of Beth's coordinates and of Alan's coordinates.

(e) Draw a space-time diagram from Carl's viewpoint, and label the worldlines of Alan, Beth, and Carl.

(f) Using Carl's measurements for Alan, repeat the steps of part (b) to get the proper time intervals. Then repeat for Carl's measurements of Beth.

(g) What does Carl compute for the proper time interval between E1 and E2 for Alan? for Beth? How do these compare with what Alan computes? Do the results agree with the Principle of Maximal Aging?

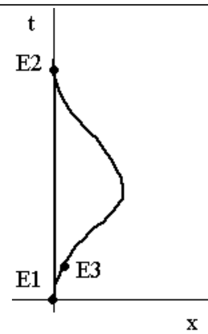


Figure 4.6: Worldlines for Alan and Beth with three events marked. How do the elapsed times between E1 and E2 compare for the two observers?

So apparently everything is self-consistent, if a bit weird. We can also make the connection to a free particle, a free particle, by Newton's first law, will have a constant velocity, and thus move along a straight worldline. So we can state the principle of maximal aging as "A free particle will travel the worldline in which it ages by the largest amount, larger than the aging if it follows a path where it is accelerated."

The Principle of Maximal Aging carries over to General Relativity where it can be used to explain the curvature of light near a massive star.

4.15 Maximal or Minimal

Spacetime Physics, Chapter 5 Section 8 points out an important area where confusion can arise.

Suppose Beth carries a laser that she flashes twice, one after the other separated by a time Δt^{Beth}. Alan watches Beth move to the right at β.

Case 1: What Alan measures for the time interval between flashes (and remember, as an intelligent observer he corrects for any transit time effects) is a time Δt^{Alan}. According to our ideas of time dilation, $\Delta t^{Alan} > \Delta t^{Beth}$ since Beth is the proper observer of the time interval, being present at both events. This is true whatever the speed of Beth (as measured by Alan), and therefore we can say that Beth measures the proper time interval and it is the *minimum* time interval.

Case 2: But we have just made a big deal about the time interval measured by an observer traveling along a straight line being the *maximum*. Which is it, minimum or maximum?

Case 1 compares time intervals measured by two different observers, only one of whom measures the proper time interval. The proper time interval is always shorter than the improper time interval, hence the proper time interval is a minimum. The ratio of the two measured time intervals is always γ.

Case 2 also compares time intervals measured by two observers. In this case both observers see $\Delta x = 0$ m, and therefore both observers measure a proper time on their watches. However their watches will disagree, and the worldline joining the two events that appears shortest on a space-time diagram will measure the longer time, with the maximum time measured by the world line that is a straight line. There is no simple expression for the ratio of time intervals.

4.16 Summary

Paradoxes are a good way to check your understaqnding of special relativity. To remove the paradox, you must make very careful use of the rules of special relativity.

On a 1D space-time diagram, **hyperbolas** with foci on the time axis connect events that all have the same space-time interval from the origin.

World lines are the trajectory of an object through space-time, and must always have an angle of magnitude less than or equal to 45° from the time-axis.

Principle of Maximal Aging: Between two events an observer that travels on a straight line on a space-time diagram will measure the largest time interval.

Chapter 5

Time-like, Space-like, Light-like Intervals, Invariant Hyperbolae, Minkowski Diagrams

5.1 Introduction

In this chapter we will look at cause-and-effect in relativity, time-like, space-like, and light-like intervals, the light cone (again), and another way to consider the space-time diagram, Minkowski diagrams.

5.2 Time-like, Space-like, and Light-like Intervals

We have discussed the invarient space-time interval, STI, which in one space dimension is

$$STI^2 = \Delta t^2 - \Delta x^2 \tag{5.1}$$

Thus if class is held on a bus and the ground based observer, Gwen, measures the start at $(t_1 = 40, x_1 = 20)$ min and the end at $(t_2 = 80, x_1 = 45)$ min, the STI is $STI = \sqrt{(80-40)^2 - (45-20)^2} = 31.2$ min. And we expect that there is some proper observer, Pete, for whom the two events occur at the same location.

What is the speed of the Pete, the proper observer, according to Gwen?

$$\beta = \Delta x/\Delta t = 25/40 = 0.625 \tag{5.2}$$

You can verify that this works by doing the Lorentz transformations, and find that the two events both occur at $x = -6.4$ min and at times 35.2 and 66.5 min. The proper time interval for the class is 31.3 min.

Now consider a second pair of events for a stick moving past you. Event 3 is the left end of the stick at your eyes, $(t_3 = 0, x_3 = 0)$ m, and Event 4 is the right end of the stick near your toes, $(t_4 = 5, x_4 = 15)$ m. When we compute the STI we get

$$STI^2 = (5)^2 - (15)^2 = -200 \tag{5.3}$$

What does it mean when the STI^2 is negative?

Here is a simpler way to get a negative STI^2: Consider two events that occur simultaneously but at different locations, say $(t = 0, x = 0)$ m and $(t = 0, x = L)$ m, Then

$$STI^2 = 0^2 - L^2 = -L^2 \tag{5.4}$$

and this is invariant for all observers.

The two events just described could refer to measuring a proper length. For an object of some non-zero length, there is no observer who can see the both ends at the same location, that is $\Delta x > 0$ for all observers. This can be made more clear by the following example.

Example Alan holds a rod with its left end at $(0, 0)$ and it right end at $(0, 10)$, that is he can simultaneously measure the two ends and find a length of 10 m for the rod. Alan sees Beth move past at $\beta = 0.80$. Since the rod is at rest with respect to Alan, he could measure the ends at any other combinations as is shown in Table 5.1.

Table 5.1: Measuring the length of a rod, Time and position in meters.

Event	Alan		Beth	
Number	t	x	t'	x'
E1	0.00	0.00	0.00	0.00
E2	0.00	10.00	-13.33	16.67
E3		10.00	0.00	
E4	5.00	0.00		
E5	5.00	10.00		
E6		10.00		
E7	20.00	0.00		
E8	20.00	10.00		
E9		10.00		

Notice that events E1, E4, and E7 refer to measuring the left end of the rod at different times, while events E2, E3, E5, E6, E8, and E9 refer to measuring the right end.

Since the rod is at rest with respect to Alan, he can ignore the times completely and compute the length of the rod as $L^{Alan} = 10 - 0 = 10$ m.

For Beth's measurements of the coordinates of the events we use the Lorentz transformations. The result is shown for E2, and you can fill in the table for E4, E5, E7, and E8.

What are events E3, E6, and E9? If Beth wants to measure the length of a moving rod, she must simultaneously (for her) measure the positions of the left and right ends. And while E1 and E2 are simultaneous to Alan, they are NOT simultaneous for Beth. Instead we must determine where the right end of the rod is simultaneous to measuring the left end of the rod, simultaneous for Beth. Look at E1. Beth measures a time of 0 m so we need to know where the right end is at this same $t^{Beth} = 0$.

We can use the Lorentz transformation,

$$\begin{aligned} t^{Beth} &= \gamma(t^{Alan} - \beta x^{Alan}) \\ 0 &= 1.6667(t^{Alan} - 0.80(10)) \\ t^{Alan} &= 8.0\,m \end{aligned} \tag{5.5}$$

and hence

$$x^{Beth} = 1.6667(10 - 0.80(8)) = 6.0\,m \tag{5.6}$$

So Beth measures the rod to have length of 6.0 m.

Compute the rest of the table, and see that regardless of when Beth makes her measurement, she determines the rod to be 6.0 m long. This of course agrees with the length contraction formula.

Intervals where $\Delta t > \Delta x$ are called *time-like*, with $STI^2 > 0$. For events separated by a time-like interval, it is always possible to find an observer who will see the events occur at the same location. Also the temporal order of the events in time is the same for all observers, E2 will always be after E1, however the relative locations are flexible, $\Delta x = x_2 - x_1$ can be positive, zero, or negative depending on the observer. The observer who sees a space interval of 0 measures a proper time interval, and this is the smallest interval that can be measured.

Intervals where $\Delta t < \Delta x$ are called *space-like*, with $STI^2 < 0$. For events separated by a space-like interval, it is always possible to find an observer who will see the events occur at the same time. Also the spatial order of the events in time is the same for all observers, E2 will always be to the right of E1. However the timing of the evnts is flexible, $\Delta t = t_2 - t_1$ can be positive, zero, or negative depending on the observer. The observer who sees a time interval of 0, i.e. measures the events simultaneously, will measures a proper length interval, and this is the largest length interval that can be measured.

Intervals where $\Delta t = \Delta x$ are called light-like. For events separated by light-like intervals, something moving at the speed of light will measure $\Delta t = \Delta x = 0$.The only things that travel at the speed of light at our current understanding of physics are photons, the quantum particles of light. Gravitons, hypothesized but never yet detected, would also travel at the speed of light. Neutrinos were long thought to move at the speed of light, but since 1998 fairly strong evidence that they move slightly slower than c emerged.

5.3 Invariant Hyperbolae

In the last chapter we discussed the invariant hyperbola. If event E1 is at the origin, then the hyperbola shows all events E2 that have the same STI. This arises from

$$STI^2 = \Delta t^2 - \Delta x^2 = Constant \qquad (5.7)$$

If E1 is at the origin, this becomes

$$t^2 - x^2 = constant \qquad (5.8)$$

For time-like intervals the constant is positive and the hyperbolae are shown in Figure 5.1(a). There are two curves, one in the past and one in the future that have the same STI. Notice that for the future hyperbola, E2 occurs after E1 for all observers, however some observers say E1 occurs to the left of E2, some say to the right, and some (proper observers) say that the two events occur at the same location.

For space-like intervals the constant is negative and the hyperbolae are shown in Figure 5.1(b). Consider event E2 on the right

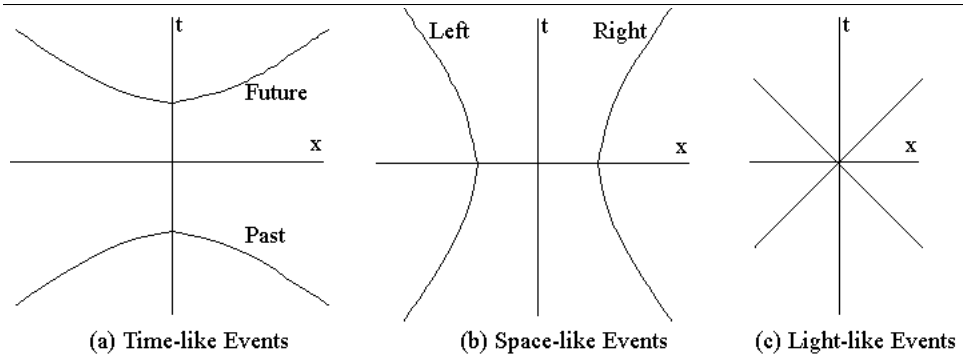


Figure 5.1: (a) Hyperbolae for time-like events, one at the origin and one on the hyperbolae. The two events are strictly ordered in time, but can occur to the left, at the same location, or to the right of each other. (b) Hyperbolae for space-like events, one at the origin and one on the hyperbolae. The two events are strictly ordered in space, but can occur before, simultaneous to, or after each other. (c) Hyperbolae (lines) for light-like events. $x = \pm t$

hyperbola. All observers will see E2 to the right of E1 (the origin), however the time order is not restricted, $t_2 > t_1 \, or \, t_2 = t_1 \, or \, t_2 < t_1$.

Figure 5.1(c) shows the hyperbolae—actually two lines—for light-like events where the constant is 0 so that $\Delta x = \pm\Delta t$.

Think about what this implies. Two events that are separated by a space-like interval have no definite time order.

For example Ethyl measures that Alan was born at $(t = 0, x = 0)$ m and Beth was born at $(t = 8, x = 10)$ m, then $STI^2 = 8^2 - 10^2 = -36\,m^2$. Can we answer the question, "Who was born first?" Certainly we can, but we will get different answers according to different observers. For Ethyl, Alan was born before Beth.

But consider Tammy who moves at $\beta = 0.80$. Applying the Lorentz transformations we get for Tammy's measurements $(t = 0, x = 0)$ m and $(t = 0, x = 6)$ m, so Tammy says that Alan and Beth were born simultaneously. Or consider Ivan who moves at $\beta = 0.95$. Applying the Lorentz transformations we get for Ivan's measurements $(t = 0, x = 0)$ m and $(t = -4.8, x = 7.7)$ m, so Ivan says that Beth was born before Alan. So there is no strict time-order to the births, however all observers will agree that Beth was born to the right of Alan.

If we change the scenario to Alan firing a gun and Beth dying of a gunshot we can see that there cannot be a cause-and-effect relation between the two events.

Notice that in our example, Tammy sees the smallest spatial separation between the two events, and for Tammy, $\Delta t = 0$ m. We can determine the speed required of Tammy as follows. We have E1 at (0, 0) m and E2 at

(t, x). For Tammy, $t^{Tammy} = 0$ so

$$t^{Tammy} = \gamma(t^{Ethyl} - \beta x^{Ethyl})$$
$$0 = \gamma(t^{Ethyl} - \beta x^{Ethyl})$$
$$t^{Ethyl} = \beta x^{Ethyl}$$
$$\beta = t^{Ethyl}/x^{Ethyl} = 8/10 = 0.8$$

or if the first event is not at the origin,

$$\beta^{space-like} = \Delta t/\Delta x \tag{5.9}$$

Wait a minute you might be saying, velocity is defined as distance over time, not time over distance. Certainly that is true, but we are not using this definition, we are using the requirement of the Lorentz transformation to get our answer.

Consider time-like intervals where $\Delta t^2 - \Delta x^2 > 0$. We can ask, "What is the velocity of a "proper" observer who would see the two events occur at the same location?" Again we first consider E1 to be at the origin. For the proper observer,

$$x' = \gamma(x - \beta t)$$
$$0 = \gamma(x - \beta t)$$
$$\beta = x/t$$

or if the first event is not at the origin,

$$\beta^{time-like} = \Delta x/\Delta t \tag{5.10}$$

Two events separated by a time-like interval could be a cause-and-effect pair. For example the pair of events E1 = me throwing a pie thrown at coordinates $(t = 0, x = 0)$ and E2 = a pie hitting you in the face at (50,30) m are separated by a time-like interval. Hence I could have been the cause of you being hit by a pie. Of course there is no guarantee that just because two events are separated by a time-like interval, that they are cause and effect—I may have thrown an apple pie, and you may have been hit by a coconut cream pie thrown by Dr. Axon.

On a space-time diagram it is easy to distinguish space-like and time-like intervals. In Chapter 4 we showed a worldline for a particle. This is repeated in Figure 5.2. This shows a particle at some definite location in time and space, plus the past history (worldline) of the particle. A light cone is centered on the current coordinates of the particle, and several other events are shown.

All events that are in the shaded regions (E2, E3, E4) are separated from the particle by space-like intervals. Event E1 is separated from the particle by a time-like interval and is an event in the particle's past. E1 could have caused an effect on the particle. Event E5 is separated from the particle by a time-like interval and is an event in the particle's future. The particle could cause an effect at E5.

As the worldline progresses through time, the (imaginary) light cone travels with it, separating space-like and time-like events from the particle.

5.4 Special Lines

In addition to the invariant hyperbolae that we have already discussed, it is helpful to discuss two other lines on a space-time diagram.

We consider the space-time diagram of Alan who sees Beth moving to the right at $\beta = 0.20$. As always, we choose the origins of Alan's and Berth's coordinate systems to

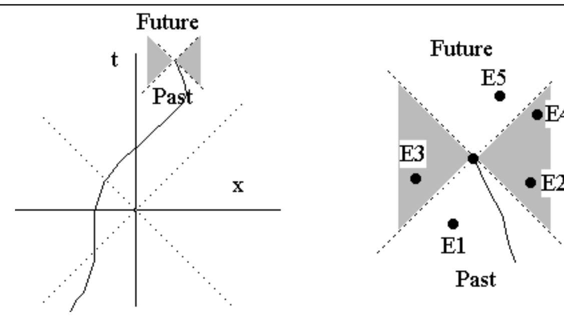


Figure 5.2: Worldline of some particle, showing the light cone and several events around the current time-space coordinate.

coincide—$x' = x = 0$ when $t' = t = 0$.

The first special line is the worldline of the origin of Beth's coordinate system, that is where Beth's origin will be at points in the future. This suggests a time-like interval, and we can use Equation 5.10 to get $x = \beta t$.

I'll try to say it in words—As Beth moves forward in time she always says that her origin is at $x' = 0$, but Alan says that her origin at time t is found at $x = \beta t$. This is shown in Figure 5.3(a). The line makes an angle α where

$$\tan \alpha = x/t = \beta \tag{5.11}$$

For $\beta = 0.20$, $\alpha = 11.3°$.

Note that for this line $x' = 0$, meaning that it is in essence the t'-axis as seen by Alan.

Now for the second special line. In words, "At $t' = 0$ Beth can measure various positions along her x' axis. What will these look like in Alan's reference frame?"

This suggests space-like intervals, and we had Equation 5.9, $t = \beta x$. This is shown in Figure 5.3(b). It also makes an angle α but with Alan's x-axis.

If this seems confusing, try this—I'll use the example numbers for $\beta = 0.20$. Here are Beth's coordinates for several points along her x' axis at $t' = 0$. I use the Lorentz transformation with $\beta' = -0.20$ to get Alan's measurements, and note that $t = \beta x$ does indeed work.

Table 5.2: Various Events for $t' = 0$

Beth		Alan	
x'	t'	x	t
0	0	0	0
1	0	1.02	0.204
2	0	2.04	0.408
3	0	3.06	0.612

Note that for the second line $t' = 0$, meaning that it is in essence the x'-axis as seen by Alan. These two special lines will form the basis of a powerful graphical way to look at special relativity, Minkowski diagrams.

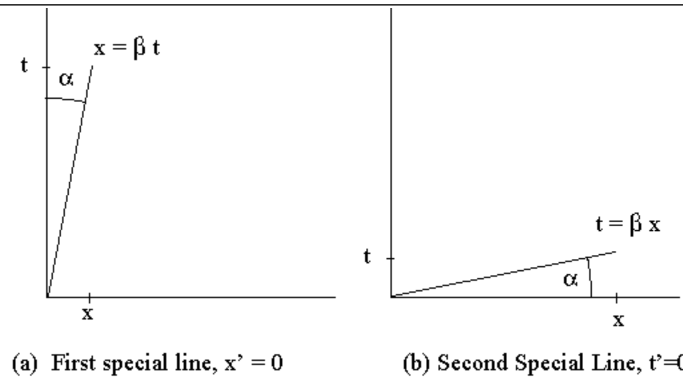


Figure 5.3: Two special lines. (a) $x' = 0$, this is the $t'-$axis. (b) $t' = 0$, this is the $x'-$axis.

5.5 Minkowski Diagrams

Minkowski diagrams are a variation of spacetime diagrams that include all reference frames on a single diagram, rather than having a different diagram for each observer.

As we saw in Section 5.4, there are two special lines that are Beth's x'- and t'-axes as viewed by Alan, the unprimed observer. The two axes make the same angle α with the respective x- and t-axes. Equation **??** tells us how to compute the angle, $\tan\alpha = \beta$.

The angle is in the range $-90° < \alpha < 90°$. As is shown in Figure 5.4, positive angles occur for Beth moving to the right, and negative for Beth moving to the left.

The remaining complication is to determine where the tick marks occur on the rotated axes.

Consider equally spaced marks on the x' axis. For all these points $t' = 0$. We can then use the Lorentz Transformations to get the coordinates in Alan's unprimed frame. Of course $(t' = x' = 0)$ transforms to $(t = 0, x = 0)$. For $(t' = 0, x' = 1)$,

$$\begin{aligned} t &= \gamma(0 - \beta' 1) = -\gamma\beta' = \gamma\beta \\ x &= \gamma(1 - \beta' 0) = \gamma \end{aligned} \quad (5.12)$$

The distance from the origin to this point is, by the Pythagorean Theorem,

$$\sqrt{(\gamma\beta)^2 + \gamma^2} = \gamma\sqrt{1 + \beta^2} \quad (5.13)$$

Example Suppose $\beta = 0.6, \gamma = 1.25$ Then the spacing on the (t', x') axes is $1.25\sqrt{1 + .6^2} = 1.46$. On the Minkowski diagram we can then place tick marks for the primed axes. This is shown on Figure 5.5.

What can the Minkowski diagram tell us? The relativity of simultaneity is obvious. Look at Figure 5.5(a). The solid and hollow dots occur at different times in Alan's frame, but at the same time in Beth's frame.

Figure 5.5(a) also shows length contraction. The solid rod is at rest in Alan's frame, with

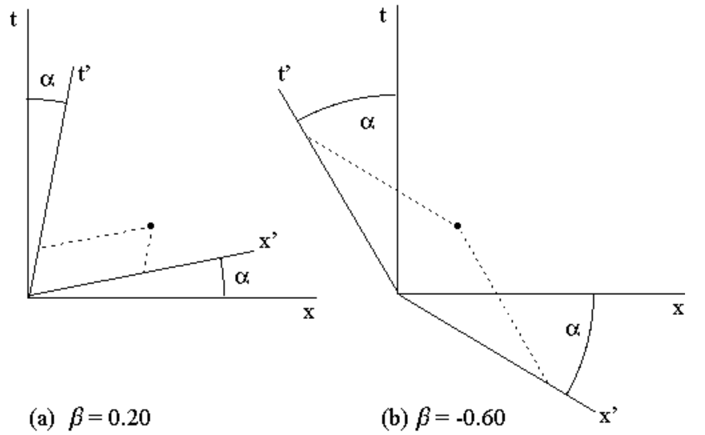


Figure 5.4: Minkowski diagrams for Beth (t', x') moving to the (a) right at 0.20, $\alpha = 11.3°$, and (b) left at -0.60, $\alpha = -31°$. Dashed lines show how to read values of (t', x').

a length of 1 unit. Since the rod is at rest with respect to him, he can measure the two ends at different times and still get a correct length. Beth sees the rod as moving, so for her to get a correct length she must measure the lengths at the same time t'—these are the hollow and solid dots at $t' = 0$. Reading the length on the x' axis gives a length of about 0.8, and that agrees with length contraction.

Figure 5.5(b) shows how the coordinates read from the diagram agree with the Lorentz transformation (no surprise, since the transformation was used to produce the diagram.) For Alan, the solid dot has coordinates $(t = 1, x = 2.5)$ m, while for Beth the coordinates are indicated with the hollow dots and are $(t = -0.64, x = 2.3)$ m by my crude measurement on my diagram. The Lorentz transformations give $(t = -0.63, x = 2.38)$, in agreement with the graphical measurement.

5.6 Summary

The relation between two events can be time-like, light-like, or space-like.

For events E1 and E2 with a time-like interval, the square of the space-time interval, $STI^2 = \Delta t^2 - \Delta x^2$ is positive and event E2 occurs after event E1 for all observers, $\Delta t > 0$. Different observers will measure different values for Δx: some will see event E2 occur to the left of E1, some will see event E2 occur to the right of E1, and one observer will see the events occur at the same location. Earlier we described this as the proper observer of the time interval, and for this proper observer the time interval is a minimum.

For events with light like intervals, the space-time interval is 0. There is no physical observer who can see the events occur at the same location or the same time. Only rays of light have this property.

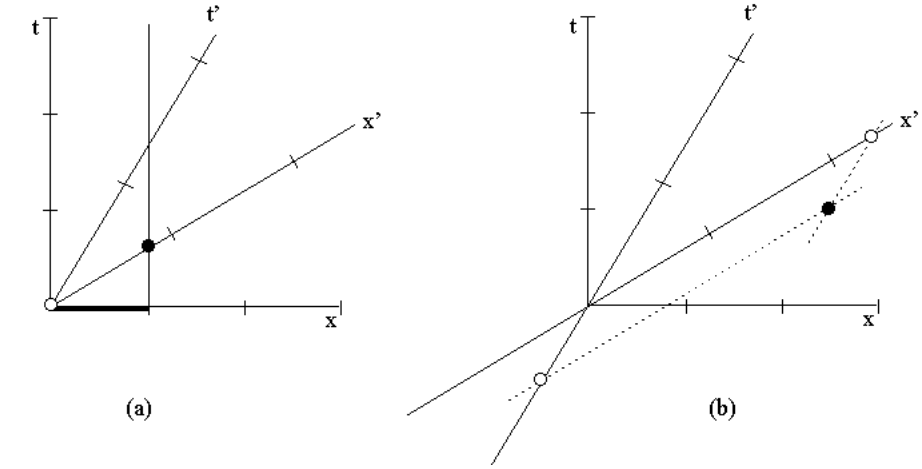


Figure 5.5: (a) Using Minkowski Diagram for Length of a rod (b) Coordinates of an event

For events with space-like intervals, the square of the space-time interval is negative, and if E2 occurs to the right of E1 for one observer, it occurs to the right for all observers, $\Delta x > 0$. Different observers will measure different values for Δt: some will see event E2 occur before E1, some will see event E2 occur after E1, and one observer will see the events occur at the same time (simultaneous).

On a space-time diagram with one event at the origin, hyperbolas with foci on the time-axis connect time-like events with the same space-time interval. Hyperbolas with foci of the space-axis connect space-like events with the same space-time interval. Diagonal lines at 45° connect light-ike events.

For an event on a world line, we can draw a "light cone" that makes it easy to categorize other events in relation to the point on the world-line as space-like, light-like, or time-like. Cause-and-effect events must be time-like (however being time-like does not imply causality.)

In regular space-time diagrams we use one diagram for each observer. Minkowski diagrams allow a single diagram to be used for different observers. If a second observer has a velocity β relative to you, the axes for that observer are given by $t = x/\beta$ for the t'-axis and by $t = \beta x$ for the x'-axis. these are rotated by and angle $\alpha = \tan^{-1}\beta$ from the t- and x- axes. Coordinates of an event in your frame are determined by projecting lines parallel to your t- and x- axes. Coordinates in the other frame are determined by projecting lines parallel to the t'- and t'-axes, but the scale is different on the two different axes.

Chapter 6

Energy and Momentum

6.1 Overview

In this final chapter we will discuss the relativistic expressions for energy, E, and momentum $\vec{p}$ and explain the meaning of perhaps the most famous physics equation of them all. $E = mc^2$. First we will discuss a formalism that is useful when dealing with space-time as well as energy-momentum, *four vectors.*

6.2 Four Vectors

First consider a **classical vector** like position, $\vec{r}$. We typically describe a vector as something having a size (magnitude) and a direction (one or more angles). Thus on a 2D surface, a position could be described as "5.0 m at a direction of 25° north of east from me."

Usually we define x- and y-axes corresponding to East and North, and then find vector components along these directions If the vector has size r and angle θ from the x-axis, then

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \tag{6.1}$$

Often in math, and sometimes in physics, a vector is written by giving the components inside parentheses like $(3, 4, -5)$.

If we have the components we can get the magnitude by

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \quad 2D \\ r &= \sqrt{x^2 + y^2 + z^2} \quad 3D \end{aligned} \tag{6.2}$$

In our parable of the Surveyors, the distance, calculated from Equation 6.2 was the invariant quantity, the same for all observers.

For relativity, the space-time interval, STI, was the invariant.

$$STI = \sqrt{\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2} \tag{6.3}$$

By parallel construction to a classical vector, a 4-vector can be written as $(\Delta t, \Delta x, \Delta y, \Delta z)$ with Equation 6.3 used to calculate the magnitude of the 4-vector[1].

[1] A second way to get the signs correct is

6.3 Classical Energy and Momentum

In high school physics you should have discussed the ideas of energy and momentum, and discussed the Laws of Conservation of Energy and of Momentum.

Energy is a scalar (simple number) that comes in many forms, kinetic energy, KE, potential energy, PE, chemical energy, E_{chem} ....

For a point object, kinetic energy is defined classically in terms of the mass and speed of the object as

$$KE = \frac{1}{2}mv^2 \quad \text{Classical Kinetic Energy} \tag{6.4}$$

The total energy of a system is defined as the sum of all types of energies,

$$E = \sum KE + \sum PE + \sum E_{chem} + \cdots \tag{6.5}$$

The Law of Conservation of Energy then says

For an isolated system of particles,
the total energy is constant for all time. (6.6)

Momentum (technically linear momentum) is a vector that is defined classically in terms of a particles mass, m and velocity $\vec{v}$ as

$$\vec{p} = m\vec{v} \tag{6.7}$$

to use $\imath = \sqrt{-1}$ and write the 4-vector as $(\imath\Delta t, \imath\Delta x, \imath\Delta y, \imath\Delta z)$ with the magnitude being $STI = \sqrt{(\Delta t^2 + (\imath\Delta x)^2 + (\imath\Delta y)^2 + (\imath\Delta z)^2)}$

Once we have a coordinate system we can determine the components of the momentum, (p_x, p_y, p_z).

For a system of particles, the total momentum is

$$\vec{p}_{tot} = \sum \vec{p}_i \tag{6.8}$$

and the Law of Conservation of Momentum is

For an isolated system of particles,
the total momentum is constant
for all time. (6.9)

Consider a one-dimensional, elastic collision between two billiard balls on a horizontal pool table shown in Figure 6.1. The first ball has mass m and initially moves to the right at $v_0 = 0.60$ towards the second ball, of mass $2m$, which is at rest. The balls collide head on and separate in a situation where only kinetic energy needs to be considered.

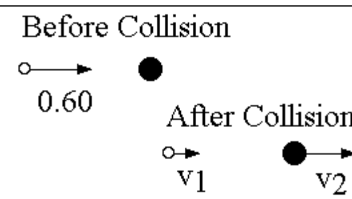


Figure 6.1: Head on 1D elastic collision of two particles

Find the speeds of the two balls after the collision.

Writing conservation of momentum,

$$
\begin{aligned}
m(0.60) + 0 &= mv_1 + (2m)v_2 \\
0.60 &= v_1 + 2v_2 \\
v_2 &= \frac{0.60 - v_1}{2}
\end{aligned}
\tag{6.10}
$$

and conservation of energy,

$$
\begin{aligned}
\frac{1}{2}m(0.60)^2 + 0 &= \frac{1}{2}mv_1^2 + \frac{1}{2}(2m)v_2^2 \\
0.36 &= v_1^2 + 2v_2^2
\end{aligned}
\tag{6.11}
$$

Now substitute Equation 6.10 into Equation 6.11 and do some algebra (and a quadratic equation) to get

$$
\begin{aligned}
v_1 = 0.60 \quad v_2 = 0 \quad \text{or} \\
v_1 = -0.20 \quad v_2 = 0.40
\end{aligned}
\tag{6.12}
$$

The first solution is our initial condition, and the second is the answer after the collision.

6.4 Does This Definition of Momentum Work Relativistically?

Another *reductio ad absurdum* proof. Assume that the solution to the collision is Section 6.3 is correct for the observer in the lab. Then by the postulates of relativity it should be correct for someone moving at constant velocity measured by the lab, say $\beta = 0.50$.

Using the velocities previously found, we use velocity transformation equations to get the velocities as measured by the moving observer. These are summarized in Table 6.1.

In the Lab frame momentum is conserved,

$$
\begin{aligned}
\text{Before} \quad m(0.600) + 2m(0) &= m(0.600) \\
\text{After} \quad m(-.200) + 2m(0.400) &= m(0.600)
\end{aligned}
\tag{6.13}
$$

The lab observer sees momentum conserved using the classical expression. However the Second Observer has a problem,

$$
\begin{aligned}
\text{Before} \quad m(0.143) + 2m(-.50) &= m(-0.857) \\
\text{After} \quad m(-0.636) + 2m(-0.125) &= m(-0.886)
\end{aligned}
\tag{6.14}
$$

The lab observer sees momentum conserved, while the moving observer says momentum is not conserved. Similar results hold true for energy conservation, total kinetic energy is conserved in teh Lab, but not for the Second Observer.

The Conservation Laws are very powerful in classical physics, and we wold like to have similar laws in special relativity. To do so we will need to redefine momentum and kinetic energy.

6.5 Dealing With Units

In Chapter 1 we found a way to have time and position have the same units. Similarly in relativity we would like to have energy and momentum have the same units. The same conversion factor c will suffice.

Classical units of energy are $(kg\ \frac{m^2}{s^2})$ or joule, J.

Before	Lab Measures	Second Observer Measures
Ball of mass m	0.60	0.143
Ball of mass $2m$	0.00	−0.50
After		
Ball of mass m	−0.20	−0.636
Ball of mass $2m$	0.40	−0.125

Table 6.1: Classical Calculation of Billiard Ball Velocities Seen in Lab and by an Observer moving at 0.50. Velocities for second observer are found using velocity transformations. While Energy and momentum are conserved for the Lab observer, they are not for the second observer as is shown in the text.

Classical units of momentum are $(kg\ \frac{m}{s})$.

By multiplying the momentum by $c = 3.00 \times 10^8$ m/s it will have the same units as energy. Or we could divide the energy by c^2 to get units of kg and divide the momentum by c to also get kg.

Example An rock has a momentum of 6.0×10^{-6} kg m/s. What is its momentum in joules?

$p = 6.0 \times 10^{-6}(3.00 \times 10^8) = 180$ J

Example A stick has energy of 1800 J and momentum of 9.0×10^{-5} kg m/s. What are these quantities in kg?

$E = 1800\ J/(3.00 \times 10^8\ m/s)^2 = 2.0 \times 10^{-14}$ kg

$p = (9.0 \times 10^{-5}\, kg\, m/s)/(3.00 \times 10^8\, m/s) = 3.0 \times 10^{-13}$ kg

The most common unit for energy and momentum used in special relativity is the electron-volt, eV. You will discus this when you take Modern Physics.

6.6 New Definitions for Momentum and Energy

Spacetime Physics has a very elegant introduction to relativistic energy and momentum—so elegant that I am not sure it is easy to learn until you already are familiar with the answer. So I will just present the relativistic formulas and discuss them.

Here are the relativistic results.

(a) Mass, m, is invariant. For example the mass of an electron is 9.11×10^{-31} kg when it is at rest or when it is moving relative to you.

(b) For a particle of mass m moving with velocity $\vec{v}$ Momentum is defined as

$$\begin{aligned} \vec{p} &= \gamma m \vec{v} \\ p_x &= \gamma m v_x \\ p_y &= \gamma m v_y \\ p_z &= \gamma m v_z \end{aligned} \tag{6.15}$$

(c) The total energy of the particle is

$$E = \gamma m c^2 \tag{6.16}$$

(d) There is an invariant quantity relating mass, energy and momentum,

$$(mc^2)^2 = E^2 - (p_x c)^2 - (p_y c)^2 - (p_z c)^2$$

Or if all quantities have the same units,

$$m^2 = E^2 - p_x^2 - p_y^2 - p_z^2 \qquad (6.17)$$

This is applied to a single particle at a time, not a system.

(e) Relativistic kinetic energy is

$$KE = E - mc^2 = mc^2(\gamma - 1) \qquad (6.18)$$

With these definitions we maintain the Laws of Conservation of Energy and Momentum.

6.7 An Example

Let's solve the same problem that we did classically in Section 6.3, shown in Figure 6.1.

Consider a collision between two billiard balls on a horizontal pool table. The first ball has mass $m = 1$ and initially moves to the right at $v_0 = 0.60$ towards the second ballof mass $2m = 2$, which is at rest. The balls collide head on and separate in a situation where only kinetic energy needs to be considered.

Find the speeds of the two balls after the collision.

Now use the equations of Section 6.6, with subscripts 0, t for before and 1, 2 for after the collision.

Conserve momentum.

$$\begin{aligned} \gamma_0(1)(v_0) + \gamma_t(2)(v_t) &= \gamma_1(1)(v_1) + \gamma_2(2)(v_2) \\ 1.25(1)(0.6) + 1(2)(0) &= \gamma_1(1)(v_1) + \gamma_2(2)(v_2) \\ 0.75 &= \gamma_1(1)(v_1) + 2\gamma_2(v_2) \end{aligned} \qquad (6.19)$$

Conserve energy.

$$\begin{aligned} \gamma_0(1) + \gamma_t(2) &= \gamma_1(1) + \gamma_2(2) \\ 1.25(1) + 1(2) &= \gamma_1(1) + \gamma_2(2) \\ 3.25 &= \gamma_1(1) + 2\gamma_2 \end{aligned} \qquad (6.20)$$

Solving the simultaneous Equations 6.19 and 6.20 is difficult since the unknown velocities are in the γs as well as appearing by themselves. Here is how I did it. First I rearranged Equation 6.20 for γ_1

$$\gamma_1 = 3.25 - 2\gamma_2 \qquad (6.21)$$

Put this into Equation 6.19

$$\begin{aligned} 0.75 &= (3.25 - 2\gamma_2)v_1 + 2\gamma_2 v_2 \\ v_1 &= \frac{0.75 - 2\gamma_2 v_2}{3.25 - 2\gamma_2} \end{aligned} \qquad (6.22)$$

Use this to write an expression for γ_1^2

$$\begin{aligned} \gamma_1^2 = \frac{1}{1 - v_1^2} &= \frac{1}{1 - \left(\dfrac{0.75 - 2\gamma_2 v_2}{3.25 - 2\gamma_2}\right)^2} \\ &= \frac{(3.25 - 2\gamma_2)^2}{(3.25 - 2\gamma_2)^2 - (0.75 - 2\gamma_2 v_2)^2} \\ &= \frac{(3.25 - 2\gamma_2)^2}{10.5625 - 13\gamma_2 + 4\gamma_2^2 - 0.5625 + 3\gamma_2 v_2 - 4\gamma_2^2 v_2^2} \\ &= \frac{(3.25 - 2\gamma_2)^2}{10 - \gamma_2(13 - 3v_2) + 4\gamma_2^2(1 - v_2^2)} \\ &= \frac{(3.25 - 2\gamma_2)^2}{10 - \gamma_2(13 - 3v_2) + 4} \\ &= \frac{(3.25 - 2\gamma_2)^2}{14 - \gamma_2(13 - 3v_2)} \end{aligned} \qquad (6.23)$$

We can equate this to the square of Equation 6.21

$$\begin{aligned}(3.25-2\gamma_2)^2 &= \frac{(3.25-2\gamma_2)^2}{14-\gamma_2(13-3v_2)}\\ 1 &= \frac{1}{14-\gamma_2(13-3v_2)}\\ 14-\gamma_2(13-3v_2) &= 1\\ \gamma_2 &= \frac{13}{13-3v_2} \qquad (6.24)\end{aligned}$$

Finally, we square both sides in express γ_2 in terms of v_2

$$\begin{aligned}\frac{1}{1-v_2^2} &= \frac{169}{169-78v_2+9v_2^2}\\ 169-78v_2+9v_2^2 &= 169(1-v_2^2) = 169-169v_2^2\\ 178v_2^2-78v_2 &= 0\\ v_2 = 0 \quad \text{or} \quad v_2 &= \frac{78}{178} = 0.438 \qquad (6.25)\end{aligned}$$

Using Equation 6.22 we find that $v_1 = 0.60$ or $v_1 = -0.220$. The first answers are the initial velocities and the second answers are the final velocities.

Now we can check to see whether these expressions maintain the Conservation Laws for an observer moving at $\beta = 0.50$.

Putting these values into Equations 6.19 and 6.20 we get the results in Table 6.3 (all quantities in the same unit).

So each observer (lab and moving) are happy because momentum is conserved and energy is conserved. They get different values for the total energy and the total momentum, but these values are not supposed to be invariant. Instead the quantity $\sqrt{E^2-p^2}$ should be invariant, see Table 6.4.

Initial Values	Lab	Moving Observer
v_0	0.600	0.143
γ_0	1.250	1.010
v_t	0.000	-0.500
γ_t	1.000	1.155
Final Values		
v_1	-0.220	-0.648
γ_1	1.025	1.313
v_2	0.438	-0.079
γ_2	1.112	1.003

Table 6.2: One Dimensional Collision of billiard balls treated relativistically. Velocities and dilation factors for Lab and Second Observer.

Indeed $\sqrt{E^2-p^2} = m$ is the invariant mass, and invariant means the same before or after a collision and for all observers.

6.8 Evidence

The most direct evidence of the validity of these equations comes from radioactive decay. Among the early pioneers of radioactivity (a term she coined) was Maria Sklodowska who became Marie Curie when she married. She chemically separated some radioactive nuclides[2], starting with several tons of uranium ore (pitchblende) and obtaining less than a milligram of a nuclide she named after her home country of Poland, Polonium.

[2]Nuclide refers to a unique combination of protons and neutrons in a nucleus. Isotopes are several nuclides that have the same number of protons but different numbers of neutrons. Thus He^4, O^{16}, Pb^{206} are three nuclides, while He^4, He^5, and He^6 are three isotopes of helium (and also three nuclides.

Initial	Lab	Moving Observer		Lab	Moving Observer
p_0	0.750	0.144	E_0	1.250	1.010
p_t	0.000	-1.154	E_t	2.000	2.309
p_{total}	0.750	-1.010	E_{tot}	3.250	3.319
Final					
p_1	-0.225	-0.851	E_1	1.025	1.313
p_2	0.975	-0.159	E_2	2.225	2.006
p_{total}	0.750	-1.010	E_{tot}	3.250	3.319

Table 6.3: Momenta and energy for relativistic collisions of two billiard balls.

	Lab			Moving		
	E	p	$m = \sqrt{E^2 - p^2}$	E	p	$m = \sqrt{E^2 - p^2}$
Mass 1 before	1.250	0.750	1.000	1.010	0.144	1.000
Mass 2 before	2.000	0.000	2.000	2.309	-1.155	2.000
Mass 1 after	1.025	-0.225	1.000	1.313	-0.852	1.000
Mass 2 after	2.225	0.975	2.000	2.006	-0.159	2.000

Table 6.4: Although energy and momentum values vary for different observers, the mass, Equation 6.17, is an invariant for all observers

Scientists can now measure the mass of the nuclides very accurately, and with these results we can see that mass is not conserved but that the relativistic energy-momentum equations do explain the process. Consider the alpha decay of the most abundant isotope of Polonium[3],

$$Po^{210} \rightarrow Pb^{206} + He^4 \qquad (6.26)$$

Careful measurements of the masses of the three nuclides, in “u”, in Equation 6.26 yield

Nuclide	Mass (u)	Uncertainty (u)
Po^{210}	209.9828737	0.0000013
Pb^{206}	205.9744653	0.0000013
He^4	4.00260325415	0.00000000006

If we check to see if mass is conserved we see that

$$\begin{aligned} 209.9828737 &\neq 205.9744653 + 4.0026033 \\ &= 209.9770686 \end{aligned} \qquad (6.27)$$

where we have rounded to the same number of digits after the decimal point.

Evidently the original nuclide has more mass than the total mass of the products of the decay. What has happened to the extra mass that we started out with? The extra mass ends up as kinetic energy of the two daughter nuclides, so we expect that the total kinetic energy of this decay will be (209.9828737 - 209.9770686) u = 0.00580515 u.

[3] This isotope is often found in antistatic brushes used in photography. It also is the isotope that killed Alexander Litvinenko in 2006 (a lethal dose is less than 1 μg!) This isotope is also used in remote power sources (e.g. the moon.)

This is using a mass unit for energy. We can easily find the conversion from u to kg, $1u = 1.66053886 \times 10^{-27}$ kilograms and convert to get a kinetic energy of $1.66053886 \times 10^{-30}$ kg, and then use $c = 299792458$ m/s to get the result in joules, 8.6637×10^{-13} J. The most common unit of energy when dealing with atoms and nuclei is the electron Volt, 1 eV $= 1.60217646 \times 10^{-19}$ J resulting in a kinetic energy of 5.407×10^{6} eV = 5.407 MeV.

The measured value for this decay is 5.407 MeV, in exact agreement.

6.9 Massless Particles

When you take Modern Physics you will discuss light not only as a wave (with wavelength and frequency) but as a particle (with momentum and energy.) When light is treated as a particle it is called a *photon*. The invariant mass of a photon is zero.

How do we treat the energy and momentum of a photon? From the relativistic expressions in Equations 6.15, 6.16, and 6.17 we see that in order for momentum to be nonzero when mass is zero, the value of γ must be infinite, and hence the speed of the particle must be exactly 1. Also for the photon Equation 6.17 tells us that

$$E_{photon} = |p_{photon}| \tag{6.28}$$

If you watch Star Trek you have heard the word positron (Data had a positronic brain). A positron has the same mass as an electron but is positively charged. When a positron meets an electron the two annihilate to produce photons, usually 2, rarely 3 or more. The symbols for the electron and positron are e^-, e^+ and for the photon γ—yes this is confusing since we have used γ for the dilation factor.

Consider the annihilation shown in Figure 6.2. An electron moving at 0.50 annihilates with a positron at rest. After the collision there are two photons moving in opposite directions. We want to find the energy and momentum of the two photons.

All quantities will be measured in the same unit, the MeV. The mass of the electron and positron is 0.511 MeV. For a speed of 0.50, $\gamma = 1.155$.

Energy Conservation yields

$$\begin{aligned} 1.155(0.511) + 1.000(0.511) &= E_1 + E_2 \\ 1.101 &= E_1 + E_2 \end{aligned} \tag{6.29}$$

Momentum conservation gives

$$\begin{aligned} 1.155(0.511)(0.50) + 0 &= p_1 + p_2 \\ 0.295 &= -|p_1| + |p_2| \\ 0.295 &= -E_1 + E_2 \end{aligned} \tag{6.30}$$

Solving the last two equations we get

$$\begin{aligned} E_1 &= |p_1| = 0.698 \text{ MeV} \\ E_2 &= |p_2| = 0.403 \text{ MeV} \end{aligned} \tag{6.31}$$

Table 6.5 summarizes the energy and momentum of the objects before and after the annihilation and shows that both energy and momentum are conserved. For the moment focus on the unprimed momenta and energy—the primed quantities will be discussed in Section 6.12.

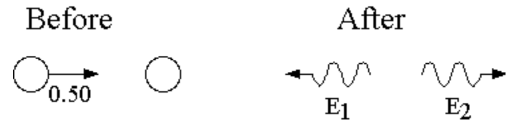


Figure 6.2: A moving electron meets positron at rest. They annihilate leaving two photons.

6.10 The Compton Effect

One final example of using energy and momentum conservation is the derivation of the Compton Formula.

In 1918 Ohio-born Arthur Compton began studying the energy of scattered X-rays. He found that the X-rays scattered off "free electrons" and was able to use the ideas of quantum mechanics and special relativity to deduce the theory. For this achievement he won the Nobel prize in physics in 1927.

Assume a photon of energy and momentum $E_1 = p_1$ moves along the x-axis toward an electron , mass m, at rest. After the collision the photon has energy and momentum $E_2 = p_2$ and moves at the angle θ while the electron has energy and momentum E and p and moves at the angle ϕ, as shown in Figure 6.3.

In the lab it is relatively easy to measure the photon energies before and after the collision, and to measure the angle through which the photon scatters. Our goal is to find an expression for the final photon energy as a function of initial photon energy, E_1, the electron mass, m, and the angle θ.

First write down Conservation of Energy.

$$\begin{aligned} E_1 + m &= E_2 + E \\ E &= E_1 + m - E_2 \end{aligned} \tag{6.32}$$

For the horizontal component of momentum, using the fact that for a photon energy equals momentum,

$$\begin{aligned} E_1 + 0 &= E_2 \cos\theta + p\cos\phi \\ p\cos\phi &= E_1 - E_2\cos\theta \end{aligned} \tag{6.33}$$

and in the y-direction

$$\begin{aligned} 0 + 0 &= E_2 \sin\theta - p\sin\phi \\ p\sin\phi &= E_2\sin\theta \end{aligned} \tag{6.34}$$

Now square Equations 6.33 and 6.34 and add, using the fact that $\sin^2\phi + \cos^2\phi = 1$ to get

$$p^2 = E_2^2 + E_1^2 - 2E_1E_2\cos\theta \tag{6.35}$$

Square Equation 6.32 to get

$$E^2 = E_1^2 + E_2^2 + m^2 - 2E_1E_2 + 2mE_1 - 2mE_2 \tag{6.36}$$

But $E^2 - p^2 = m^2$, and putting Equations 6.36 and 6.35 into this and doing some algebra yields

$$-E_1E_2 + mE_1 - mE_2 + E_1E_2\cos\theta = 0 \tag{6.37}$$

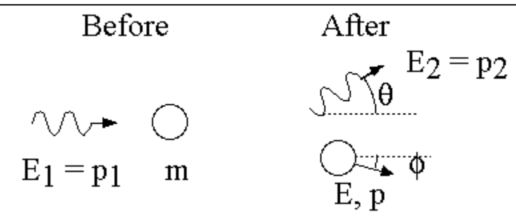


Figure 6.3: Compton Effect: Scattering of a photon by an electron through an angle θ .

and with a bit more algebra this can be written in the form

$$\frac{m}{E_2} - \frac{m}{E_1} = 1 - \cos\theta \tag{6.38}$$

Compton measured wavelength rather than energy, using the quantum mechanical relation (also from Einstein) $E = hc/\lambda$ and wrote the equation (in SI units, with the c's) as

$$\lambda_2 - \lambda_1 = \frac{h}{mc}(1 - \cos\theta) \tag{6.39}$$

The change in wavelength is very small, since $h/mc = 2.42630994 \times 10^{-12}$ m and X-rays have wavelengths in the range of 1×10^{-8} to 1×10^{-11} m. The effect is unmistakable when we use gamma-ray photons with shorter wavelengths, such as a gamma ray from the decay of a common nuclide, cobalt-60 with wavelength 1.2×10^{-12} m.

6.11 Low Speed Approximations

For 300 years the expressions of classical mechanics worked perfectly fine to explain energy and momentum. Let's chsck to see if the relativistic equations become the same as classical at low speeds.

Momentum

At low speed $\gamma = 1$ and the expression for momentum reduces as

$$p = \gamma m v \rightarrow m v \tag{6.40}$$

which is the classical expression.

Kinetic energy in relativity is defined as $E - m$. Does this approach our classical limit?

$$\begin{aligned} KE &= E - m \\ &= \gamma m - m \\ &= m\left((1 - v^2)^{-1/2} - 1\right) \\ &\approx m\left(1 + \left(\frac{-1}{2}\right)(-v^2) - 1\right) \\ &\approx \frac{1}{2} m v^2 \end{aligned} \tag{6.41}$$

Success! At low speeds the relativistic expression becomes the classical expression.

6.12 Lorentz Transformations

Earlier we had the Lorentz Transformations for time and position. The transformations for energy and momentum are similar, and relate the measurements of energy and momentum recorded by one observer to the energy and momentum recorded by a second observer moving along the x-axis at velocity β.

$$\begin{aligned} E' &= \gamma(E - \beta p_x) & E &= \gamma(E' - \beta' p'_x) \\ p'_x &= \gamma(p_x - \beta E) & p_x &= \gamma(p'_x - \beta' E') \\ p'_y &= p_y & p'_z &= p_z \end{aligned} \tag{6.42}$$

Example Consider the positron-electron annihilation of Section 6.9 as viewed by an observer moving to the right at $\beta = 0.267946$, $\gamma = 1.03795389$. What does this observer measure for the energy and momentum of the two photons?

Table 6.5 shows the results of the original calculations and the Lorentz transformations. Notice that although the two observers see different values for the total energy and the total momentum, they both see that momentum and energy conservation work.

6.13 Electric and Magnetic Fields

The culmination of special relativity is the description of electric and magnetic fields. Here I will just describe the invariant quantities and give the Lorentz Transformations. This may be covered in a junior level E&M course.

Michael Fowler lays out the basic issue at `http://galileo.phys.virginia.edu/classes/252/rel\_el\_mag.html`. A neutral wire carries a current of moving electrons, all having a velocity v to the right. Outside the wire is a positive charge, also moving right and with the same speed v.

In this frame there is no electric force, but there is a magnetic force on the charge, $\vec{F} = q\vec{v} \times \vec{B}$.

Imagine the view of an observer moving at the same velocity as the external charge. For her, the external charge is at rest, the electrons are at rest, and the positive charge in the wire moves to the left at speed v. There is no magnetic force, and a classical view of the wire would suggest no electric force since the wire is still expected to be neutral.

This suggests a paradox that can be resolved by considering length contraction. In the correct analysis the wire is no longer neutral in the second reference frame and there is an electric force on the external charge.

Just as with space/time, and energy/momentum, the equations are easier if we use the same units for both quantities. In the SI system electric field is in units of (N/C) while magnetic field is in units of tesla, T. To get the two fields in the same units, (N/C), multiply the magnetic field by our conversion factor c. In the equations below I assume the same units for E and B.

Here are the Lorentz Transformations for electric and magnetic fields—notice that they are similar but not identical to our other Lorentz Transformations.

Before the Annihilation (MeV)			
	Electron	Positron	Total
p_x	0.295026	0.000000	0.295026
p'_x	0.142121	-0.142117	0.000004
E	0.590052	0.511000	1.101052
E'	0.530395	0.530394	1.060790
After the Annihilation (MeV)			
	Photon γ_1	Photon γ_2	Total
p_x	-0.403013	0.698039	0.295026
p'_x	-0.07106	0.07106	0.00000
E	0.403013	0.698039	1.101052
E'	0.530395	0.530395	1.060790

Table 6.5: Energy and Momentum in Positron Annihilation. In the unprimed system an electron moving at 0.50 toward a stationary positron annihilates to produce two photons. The primed values are tose seen by an observer moving to the right at 0.267946.

$$E'_x = E_x \qquad B'_x = B_x \tag{6.43}$$

$$E'_y = \gamma(E_y - \beta B_z) \qquad B'_y = \gamma(B_y + \beta E_z) \tag{6.44}$$

$$E'_z = \gamma(E_z + \beta B_y) \qquad B'_z = \gamma(B_z - \beta E_y) \tag{6.45}$$

For space/time and momentum/energy we look for invariant quantities. For electric and magnetic phenomena the invariants are

- Charge
- $E^2 - B^2$
- $\vec{E} \cdot \vec{B} = E_x B_x + E_y B_y + E_z B_z$

All applications of these will be left to future courses.

6.14 Summary

In order to keep the Law of Conservation of Momentum and the Law of Conservation of Energy valid, we must redefine some quantities.

It is useful to have the same unit for energy, mass, and momentum. If we use our traditional unit of energy, the conversion factors for momentum and mass are

- Mass in units of energy = mc^2 where m is the mass in units like kg.
- Momentum in units of energy = pc where p is in units like kg m/s.

Relativistic momentum is defined as $\overrightarrow{p} = \gamma m \overrightarrow{v}$ where we can treat p as having units of kg m/s providing v is in m/s and m is in kg, or we can treat p and m as having relativistic units of energy, wth v being the dimensionless velocity.

Relativistic total energy is defined as $E =$

γm if m is in relativistic energy units, or $E = \gamma mc^2$ for m in kg.

With these definitions the Laws of Conservation of Momentum and Energy still work.

The quantity $m^2 = E^2 - p^2$ (relativistic units) or $(mc^2)^2 = E^2 - (pc)^2$ (SI units) is an invariant quantity, the same for all observers.

Kinetic energy is defined in relativistic units as $KE = E - m = (\gamma - 1)m$

In the limit of low speeds the expressions for momentum and kinetic energy revert to the classical expressions.

Relativity allows for the existence of particles with no mass, with the only experimentally observed such creature being the photon. Massless particles travel at the speed of light, and for them $E = p$.

Two observers will measure different quantities for energy and momentum. The quantities are related by Lorentz transformations. As always we assume that the observers move along the x-axis.

$$\begin{aligned} E' &= \gamma(E - \beta p_x) \quad & E &= \gamma(E' - \beta' p'_x) \\ p'_x &= \gamma(p_x - \beta E) \quad & p_x &= \gamma(p'_x - \beta' E') \\ p'_y &= p_y \quad & p'_z &= p_z \end{aligned} \tag{6.46}$$

More complicated transformation equations are available for electric and magnetic fields.